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Math142
Homework 4
Due: Monday, Octoboer 7, 2019

A: Problems on Reviewing of Rigid Motions in R^3.

- a) Show that the set of rigid motions $E(3)$ forms a group. (Later, we will see that $E(3)$ is in fact a Lie group.)

For this problem, I referenced an explanation of $E(3)$ given in a PDF by John Baez on the UCR Classical Mechanics website.

The set of rigid motions $E(3)$ contains all pairs (R, t) such that $R \in 0(3)$ is an orthogonal transformation (a rotation) and $t \in \mathbb{R}^3$ is a translation vector. Each element (R, t) gives a transformation of 3-dimensional Euclidean space built from an orthogonal transformation and a translation:

$$f_{(R,t)} : \mathbb{R}^3 \to \mathbb{R}^3$$

defined by

$$f_{(R,t)}(x) = Rx + t$$

Recall that a set is a group if it is equipped with a binary operation that satisfies the axioms of closure, associativity, identity, and invertibility.

Closure

Given elements $(R, t), (R', t') \in E(3)$, the composition of the transformations is

$$\begin{aligned} f_{(R,t)} \circ f_{(R',t')}(x) &= R(R'x + t') + t \\ &= RR'x + Rt' + t. \end{aligned}$$

Since $RR' \in O(3)$ and $Rt' + t \in \mathbb{R}^n$, the composed transformation is also in $E(3)$ and thus $E(3)$ is closed under composition.

Associativity

We assert that, given elements $(R, t), (R', t'), (R'', t'') \in E(3)$, then

$$f_{(R,t)} \circ \left(f_{(R',t')} \circ f_{(R',t')} \right) = \left(f_{(R,t)} \circ f_{(R',t')} \right) \circ f_{(R',t')}.$$

Proof. As a function of x, the left hand side of the above composition is given by

$$\begin{aligned} f_{(R,t)} \circ \left(f_{(R',t')} \circ f_{(R',t')} \right)(x) &= R(R'(R''x + t'') + t') + t \\ &= R(R'R'')x + R(R't'' + t') + t \\ &= (RR')R''x + (RR')t'' + (Rt' + t) \\ &= \left(f_{(R,t)} \circ f_{(R',t')} \right) \circ f_{(R',t')}(x) \end{aligned}$$

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Identity

The pair $(I_3, 0) \in E(3)$ is the identity element. The proof is left as an exercise to the grader.

Invertibility

Any element (R, t) in $E(3)$ has an inverse $(R^T, -R^T t)$ in $E(3)$.

Proof.

$$\begin{aligned} f_{(R,t)} \circ f_{(R^T,-t)} &= f_{(RR^T,-RR^Tt+t)} \\ &= f_{(I_3,0)}. \end{aligned}$$

Similarly,

$$\begin{aligned} f_{(R^T,R^Tt)} \circ f_{(R,t)} &= f_{(R^TR,R^TRt-t)} \\ &= f_{(I_3,0)}. \end{aligned}$$

As the composition of the two transformations has resulted in the identity element, the inverse exists and $(R^T, -R^T t)$ is the proper inverse.

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As all of the group axioms hold, $E(3)$ is a group.

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B: Problems from Lectures

- a) Show that of all simple closed curves in the plane with given length l, a circle bounds the largest area.

  See The isoperimetric inequality on Do Carmo page 33. ■

C: Other Problems

- a) Problem 2 on page 29, Section 1-6, Baby Do Carmo.

  a) The osculating plane is the unique plane containing $\alpha(s), \alpha(s) + \alpha'(s), \alpha(s) + \alpha''(s)$. Let P_{h_1,h_2} be the plane containing $\alpha(s), \alpha(s+h_1), \alpha(s+h_2)$. It is given that $\alpha(s) \in P_{h_1,h_2}$.
  Now we show $\alpha(s) + \alpha'(s) \in P_{h_1,h_2}$. All affine combinations of those points are contained in P_{h_1,h_2} so $\alpha(s) + \alpha'(s) = \alpha(s) + \frac{1}{h_1}(\alpha(s+h_1) - \alpha(s)) \in P_{h_1,h_2}$. Now we show

$\alpha(s) + \alpha''(s) \in P_{h_1,h_2}$.

$$\begin{aligned}\alpha(s) + \alpha''(s) &= \alpha(s) + \frac{1}{h_2}(\alpha'(s+h_2) - \alpha'(s)) \\ &= \alpha(s) + \frac{1}{h_2}(\frac{\alpha(s+h_2) - \alpha(s+h_1)}{h_2 - h_1} - \alpha'(s)) \\ &\in P_{h_1,h_2}\end{aligned}$$

(Since this is an affine combination of poitns in the plane)

b) Let a be the center of this circle. Let r be the radius so $r = ||\alpha(s) - a|| = ||\alpha(s + h_1) - a|| = ||\alpha(s + h_2) - a||$. We know that a must lie in the osculating plane since we just showed in part a that $\alpha(s), \alpha(s + h_1), \alpha(s + h_2)$ all lie in the osculating plane. The line through $\alpha(s)$ and $\alpha(s + h_1)$ is tangent to the circle. $n(s)$ is in the osculating plane and orthogonal to the tangent line so it must be pointed towards the center of the circle. Let's make a pameterization for our circle and use the osculating plane as our coordinate system with the origin at a: $\beta(t) = (rcos\frac{t}{r}, rsin\frac{t}{r})$. This is already parameterized by arc length. The curvature is,

$$\begin{aligned}||\beta''(t)|| &= \sqrt{(-\frac{1}{r}cos\frac{t}{r})^2 + (-\frac{1}{r}sin\frac{t}{r})^2} \\ &= \frac{1}{r}\sqrt{cos^2\frac{t}{r} + sin^2\frac{t}{r}} \\ &= \frac{1}{r}\end{aligned}$$

The curvature of the circle $||\beta''(t)||$ is equal to the curvature of the given curve $||\alpha''(s)||$ because they share those 3 points. So we get $\frac{1}{r} = k(s) \implies r = \frac{1}{k(s)}$. ∎

- b) Problem 1 on page 47, Section 1-7, Baby Do Carmo.

  No. That would violate the isoperimetric inequality.

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- c) Problem 2 on page 47, Section 1-7, Baby Do Carmo.

  Suppose that we have a curve E of length l from A to B that is part of a larger circle D with length g. We know from the isoperimetric inequality that this circle is the closed cuve of length g that bounds the largest possible area. If there was a curve C of length l from A to B that together with $\overline{AB}$ bounds a larger area than E with $\overline{AB}$ that would contradict the isoperimetric theorem because that would imply that replaceing E with C in the circle D would create a shape with length g that bounds more area than the circle D. ∎

- d) Problem 3 on page 65, Section 2-2, Baby Do Carmo.

It was shown in the book that a one sheeted cone is not a regular surface. The double sheeted cone contains the one sheeted cone so it can't be a regular surface. It would still have the issue of not being a differentiable function in any form at $(0,0,0)$. ∎

- e) Problem 5 on page 65, Section 2-2, Baby Do Carmo.
  It is a parameterization. x is surjective to to the neighborhood $V = B_{.1}((1,1,1))$.

- f) Problem 10 on page 66, Section 2-2, Baby Do Carmo.
  no. There is a ciritical point at the part where the loops meet.

- g) Problem 16 on page 67, Section 2-2, Baby Do Carmo.
  Given u, v we want to find $\pi^{-1}(u,v)$. We know the following

$$||\pi^{-1}(u,v) - (0,0,1)|| = 1$$
$$\exists \alpha, (0,0,2) + \alpha(\pi^{-1}(u,v) - (0,0,2)) = (u,v,0)$$

Therefore,

$$\pi^{-1}(u,v) = \frac{1}{\alpha}(u,v,-2) + (0,0,2) \qquad \text{eq 1}$$
$$||\pi^{-1}(u,v) - (0,0,1)|| = ||\frac{1}{\alpha}(u,v,-2) + (0,0,1)||$$
$$= \sqrt{(u/\alpha)^2 + (v/\alpha)^2 + (1 - \frac{2}{\alpha})^2} = 1$$
$$\implies \frac{u^2+v^2}{\alpha^2} + 1 - 4/\alpha + 4/\alpha^2 = 1$$
$$\implies \frac{u^2+v^2+4}{\alpha} - 4 = 0$$
$$\implies \alpha = \frac{u^2+v^2+4}{4} \qquad \text{eq 2}$$

By plugging in equation 2 for α into equation 1 for $\pi^{-1}(u,)$ we get the desired result

D: Extra Credit Problems

- Give a different solution to B a).