Joseph Gardi
Differential Geometry
Homework 5

Read:

- Baby Do Carmo, Differential Geometry of Curves and Surfaces: Sections 2-2, 2-3, 2-4 and Appendix (starting on page 118) on A Brief Review of Continuity and Differentiability
- Handouts 6 and 7
- Lecture Notes

Do:

Remember, the problems marked with an asterisk have hints in the back of the book. Additionally, many of these problems ask that you re-prove something that do Carmo proves in the reading.

A: Problems on Reviewing of Continuity and Differentiability

a) Prove the proposition 7 on page 127, Baby Do Carmo.

DEFINITION 1. *Let* $F: U \subset R^n \to R^m$ *be a differentiable map. To each* $p \in U$ *we associate a linear map* $dF_p: R^n \to R^m$ *which is called the* differential *of* F *at* p *and is defined as follows. Let* $w \in R^n$ *and let* $\alpha: (-\epsilon, \epsilon) \to U$ *be a differentiable curve such that* $\alpha(0) = p$, $\alpha'(0) = w$. *By the chain rule, the curve* $\beta = F \circ \alpha: (-\epsilon, \epsilon) \to R^m$ *is also differentiable*. Then (Fig. A2-5)

$$dF_p(w) = \beta'(0).$$

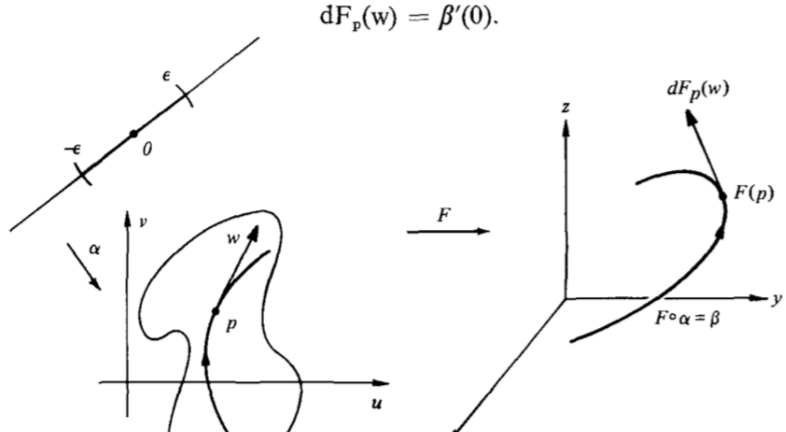

Figure A2-5

PROPOSITION 7. *The above definition of* dF_p *does not depend on the choice of the curve which passes through* p *with tangent vector* w, *and* dF_p *is, in fact, a linear map.*

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b) Prove the proposition 8 on page 129, Baby Do Carmo.

PROPOSITION 8 (The Chain Rule for Maps). *Let* $F: U \subset R^n \longrightarrow R^m$ *and* $G: V \subset R^m \longrightarrow R^k$ *be differentiable maps, where* U *and* V *are open sets such that* $F(U) \subset V$. *Then* $G \circ F: U \longrightarrow R^k$ *is a differentiable map, and*

$$d(G \circ F)_p = dG_{F(p)} \circ dF_p, \qquad p \in U.$$

Proof. The fact that $G \circ F$ is differentiable is a consequence of the chain rule for functions. Now, let $w_1 \in R^n$ be given and let us consider a curve $\alpha: (-\epsilon_2, \epsilon_2) \longrightarrow U$, with $\alpha(0) = p$, $\alpha'(0) = w_1$. Set $dF_p(w_1) = w_2$ and observe that $dG_{F(p)}(w_2) = (d/dt)(G \circ F \circ \alpha)|_{t=0}$. Then

$$d(G \circ F)_p(w_1) = \frac{d}{dt}(G \circ F \circ \alpha)_{t=0} = dG_{F(p)}(w_2) = dG_{F(p)} \circ dF_p(w_1).$$

Q.E.D.

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c) Rewrite Example 11 on page 132 of Baby Do Carmo and explain clearly why the Inverse Function Theorem (page 131) is true only in a neighborhood of a point p.

Example 11. Let $F: R^2 \longrightarrow R^2$ be given by

$$F(x, y) = (e^x \cos y,\ e^x \sin y), \qquad (x, y) \in R^2.$$

The component functions of F, namely, $u(x, y) = e^x \cos y$, $v(x, y) = e^x \sin y$, have continuous partial derivatives of all orders. Thus, F is differentiable.

It is instructive to see, geometrically, how F transforms curves of the xy plane. For instance, the vertical line $x = x_0$ is mapped into the circle $u = e^{x_0} \cos y$, $v = e^{x_0} \sin y$ of radius e^{x_0}, and the horizontal line $y = y_0$ is mapped into the half-line $u = e^x \cos y_0$, $v = e^x \sin y_0$ with slope $\tan y_0$. It follows that (Fig. A2-7)

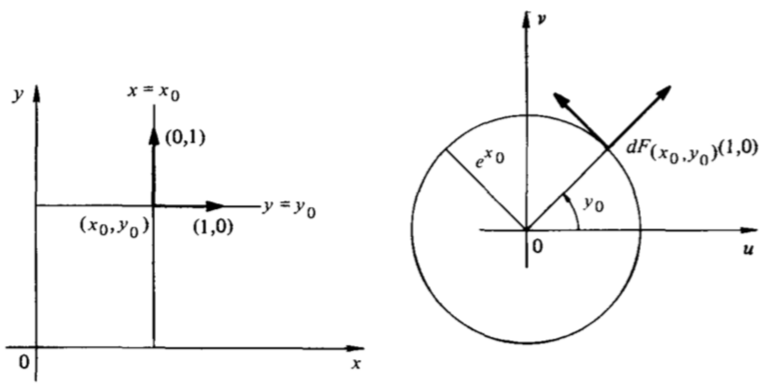

Figure A2-7

$$\begin{aligned} dF_{(x_0, y_0)}(1, 0) &= \frac{d}{dx}(e^x \cos y_0,\ e^x \sin y_0)\big|_{x=x_0} \\ &= (e^{x_0} \cos y_0,\ e^{x_0} \sin y_0), \\ dF_{(x_0, y_0)}(1, 0) &= \frac{d}{dy}(e^{x_0} \cos y,\ e^{x_0} \sin y)\big|_{y=y_0} \\ &= (-e^{x_0} \sin y_0,\ e^{x_0} \cos y_0). \end{aligned}$$

This can be most easily checked by computing the Jacobian matrix of F,

$$dF_{(x,y)} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix},$$

and applying it to the vectors $(1, 0)$ and $(0, 1)$ at (x_0, y_0).

We notice that the Jacobian determinant $\det(dF_{(x,y)}) = e^x \neq 0$, and thus dF_p is nonsingular for all $p = (x, y) \in R^2$ (this is also clear from the previous geometric considerations). Therefore, we can apply the inverse function theorem to conclude that F is locally a diffeomorphism.

Observe that $F(x, y) = F(x, y + 2\pi)$. Thus, F is not one-to-one and has no global inverse. For each $p \in R^2$, the inverse function theorem gives neighborhoods V of p and W of $F(p)$ so that the restriction $F: V \longrightarrow W$ is a diffeomorphism. In our case, V may be taken as the strip $\{-\infty < x < \infty, 0 < y < 2\pi\}$ and W as $R^2 - \{(0, 0)\}$. However, as the example shows, even if the conditions of the theorem are satisfied everywhere and the domain of definition of F is very simple, a global inverse of F may fail to exist.

INVERSE FUNCTION THEOREM. *Let* $F: U \subset R^n \longrightarrow R^n$ *be a differentiable mapping and suppose that at* $p \in U$ *the differential* $dF_p: R^n \longrightarrow R^n$ *is an isomorphism. Then there exists a neighborhood* V *of* p *in* U *and a neighborhood* W *of* $F(p)$ *in* R^n *such that* $F: V \longrightarrow W$ *has a differentiable inverse* $F^{-1}: W \longrightarrow V$.

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d) Show that an infinite cylinder after deleting a vertical line is diffeomorphic to a plane.

Let r be the radius of the cylinder and put the center of the cylinder at the origin.

Let the plane be $span([1,0,0],[0,1,0])$.
Let's use something like cylindrical coordinates. We are parameterizing the infinite cylinder with α : $(\theta, h) \to (r\cos\theta, r\sin\theta, h)$. Proving that this is a parameterization is left as an excersise.

Let $x : (a,b) \to (2\pi\frac{|a|}{|a|+1}, b)$.
I want to show that $\alpha \circ x$ is a difeomorphism between the plane and the cylinder. To do this it is sufficient to show that x is diffeomorphic since is a parameterization . It is left as an excersise to show that x is a bijection. Now to show that it is differentiable and has differentiable inverse we show that the jacobian is invertable at all points in the plane. Let $(a,b,c) \in \mathbb{R}^3$ be given.
The jacobian is,

$$\begin{bmatrix} \frac{\partial x_1}{\partial a} & \frac{\partial x_1}{\partial b} \\ \frac{\partial x_2}{\partial a} & \frac{\partial x_2}{\partial b} \end{bmatrix}$$
$$= \begin{bmatrix} 2\pi \cdot \frac{1-|a|/(|a|+1)}{|a|+1} & 0 \\ 0 & 1 \end{bmatrix}$$

Now we just need to show this matrix is invertable for all a. The determinant is

$$2\pi \cdot \frac{1-|a|/(|a|+1)}{|a|+1}$$

The determinant approaches zero but never actually reaches it so x is diffeomorphic. ■

B: Problems from Lectures

a) Use Inverse Function Theorem to give a proof of proposition 2, page 59, Baby Co Carmo.

PROPOSITION 2. *If* $f: U \subset R^3 \to R$ *is a differentiable function and* $a \in f(U)$ *is a regular value of* f, *then* $f^{-1}(a)$ *is a regular surface in* R^3.

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b) Use Inverse Function Theorem to give a proof of proposition 4, page 64, Baby Co Carmo.

PROPOSITION 4. *Let* $p \in S$ *be a point of a regular surface* S *and let* $\mathbf{x}: U \subset R^2 \to R^3$ *be a map with* $p \in \mathbf{x}(U)$ *such that conditions* 1 *and* 3 *of Def.* 1 *hold. Assume that* $\mathbf{x}$ *is one-to-one. Then* $\mathbf{x}^{-1}$ *is continuous.*

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C: Other Problems

a) Problem 7 on page 66, Section 2-2, Baby Do Carmo.

7. Let $f(x, y, z) = (x + y + z - 1)^2$.

a. Locate the critical points and critical values of f.

b. For what values of c is the set $f(x, y, z) = c$ a regular surface?

c. Answer the questions of parts a and b for the function $f(x, y, z) = xyz^2$.

First we find the set of critical points $C = \{(x,y,z) \in \mathbb{R}^3 : \mathbf{d}f(x,y,z) = 0\}$ So for each $(x,y,z) \in C$,

$$\begin{aligned} &\mathbf{d}f(x,y,z) = 0 \\ \iff &\begin{bmatrix} 2(x+y+z-1) & 2(x+y+z-1) & 2(x+y+z-1) \end{bmatrix} = 0 \\ \iff &x+y+z-1 = 0 \end{aligned}$$

This is the equation of a plane. So C is the set of points in a plane. The critical values are the image $f(C) = \{f(x,y,z) : (x,y,z) \in C\} = \{(x+y+z-1)^2 : x+y+z-1 = 0\} = \{0\}$ ■

b) Problem 11 on page 66, Section 2-2, Baby Do Carmo.

11. Show that the set $S = \{(x, y, z) \in R^3; z = x^2 - y^2\}$ is a regular surface and check that parts a and b are parametrizations for S:

a. $\mathbf{x}(u, v) = (u + v, u - v, 4uv)$, $(u, v) \in R^2$.

***b.** $\mathbf{x}(u, v) = (u \cosh v, u \sinh v, u^2)$, $(u, v) \in R^2$, $u \neq 0$.

Which parts of S do these parametrizations cover?

I'll do both part a and part b at once. To show that a, b are reegular you can show that the differential for both functions is invertable. So for a the differential is,

$$\mathbf{d}a(u,v) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 4v & 4u \end{bmatrix}$$

This matrix is an invertable map everywhere because it has the minor $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ which is not invertable. Similarly, for b,

$$\mathbf{d}b(u,v) = \begin{bmatrix} \cosh v & u \sinh v \\ \sinh v & u \cosh v \\ 2u & 0 \end{bmatrix}$$

It was given that $u \neq 0$ for inputs to b so $2u$ is not a multiple of 0. Therefore, the columns of $\mathbf{d}b$ are always linearly independent. So the map is always invertable.

Now let's show that the images of both functions are contained in S. let's call the functions a, b rather than calling both of them x. For all $p = (u+v, u-v, 4uv) \in x(\mathbb{R}^2)$, $(u+v)^2 - (u-v)^2 = u^2 + 2uv + v^2 - (u^2 - 2uv + v^2) = 4uv$. So $z = x^2 - y^2$ is satisified for at p. Therefore $p \in S$. Similarly for b:

$$\forall p = (u \cosh v, u \sinh v, u^2) \in image(b),$$
$$(u \cosh v)^2 - (u \sinh v)^2 = u^2(\cosh^2 v - \sinh^2 v) = u^2$$

To show that a, b are homeomorphic we have to show they are bijective. First I do it for a. Suppose that $a(u_1, v_1) = a(u_2, v_2)$. Then I will show that $u_1, = u_2, v_1, = v_2$. This gives us the matrix equation,

$$A \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = A \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}$$
$$\implies \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \text{(because } A \text{ is invertible)}$$
$$\text{where } A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Showing b is bijective is left as an excersise. Then to show that a, b are homeomorphic we observe that they are continuous and have continuous inverses. To Show that the x covers $V \bigcap S$ for some neighborhood $V \subset S$ just . ■

c) Problem 1 on page 80, Section 2-3, Baby Do Carmo.

***1.** Let $S^2 = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$ be the unit sphere and let $A: S^2 \longrightarrow S^2$ be the (*antipodal*) map $A(x, y, z) = (-x, -y, -z)$. Prove that A is a diffeomorphism.

First observe it is a bijection. Then observe that the jacobian is inverable everywhere. ■

d) Problem 8 on page 80, Section 2-3, Baby Do Carmo.

***8.** Let $S^2 = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$ and $H = \{(x, y, z) \in R^3; x^2 + y^2 - z^2 = 1\}$. Denote by $N = (0, 0, 1)$ and $S = (0, 0, -1)$ the north and south poles of S^2, respectively, and let $F\colon S^2 - \{N\} \cup \{S\} \longrightarrow H$ be defined as follows: For each $p \in S^2 - \{N\} \cup \{S\}$ let the perpendicular from p to the z axis meet $0z$ at q. Consider the half-line l starting at q and containing p. Then $F(p) = l \cap H$ (Fig. 2-20). Prove that F is differentiable.

Given a point $p = (x, y, z) \in S^2$ we find q by projection onto the z-axis, so $q = (0, 0, z)$. The half line joining q to p is parameterized by (tx, ty, z) where $0 \leq t$. This line intersects H when

$$t^2x^2 + t^2y^2 - z^2 = 1$$

solving for t we get

$$t = \frac{\sqrt{1+z^2}}{\sqrt{x^2+y^2}}$$

So

$$F(p) = \left(\frac{\sqrt{1+z^2}}{\sqrt{x^2+y^2}}x, \frac{\sqrt{1+z^2}}{\sqrt{x^2+y^2}}y, z \right)$$

Let $V = \mathbb{R}^3 - \{(x, y, z) \mid x = y = 0\}$, then V is an open subset of $\mathbb{R}^3$ and F has continuous partial derivatives on V. Therefore, F is differentiable on V

Since $S^2 - (\{N\} \cup \{S\}) \subset V$ and S^2 and H are regular surfaces, we have, by Example 3 of section 2-3, that $F|_{S^2}\colon S^2 \to H$ is differentiable. ∎

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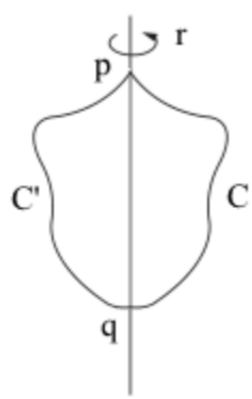
e) Problem 10 on page 81, Section 2-3, Baby Do Carmo.

10. Let C be a plane regular curve which lies in one side of a straight line r of the plane and meets r at the points p, q (Fig. 2-21). What conditions should C satisfy to ensure that the rotation of C about r generates an extended (regular) surface of revolution?

Figure 2-21

The surface generated needs have parametrizations at the points p and q. In addition the curve C should have no self intersections. These conditions will be meet if the curve formed by joining C with its reflection over r is a simple closed regular curve (see image below).

More formally, let C' be the curve given by the reflection of C over the line r. We require that the curve C satisfy the condition that $C \cup C'$ is a simple regular closed curve.



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f) Problem 12 on page 81, Section 2-3, Baby Do Carmo.

12. Parametrized surfaces are often useful to describe sets Σ which are regular surfaces except for a finite number of points and a finite number of lines. For instance, let C be the trace of a regular parametrized curve $\alpha : (a, b) \longrightarrow R^3$ which does not pass through the origin $O = (0, 0, 0)$. Let Σ be the set generated by the

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f continued)

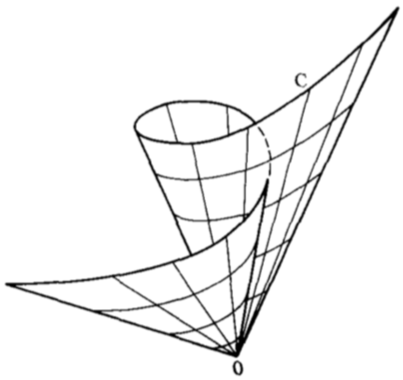

Figure 2-22

displacement of a straight line l passing through a moving point $p \in C$ and the fixed point 0 (a cone with vertex 0; see Fig. 2-22).

a. Find a parametrized surface **x** whose trace is Σ.

b. Find the points where **x** is not regular.

c. What should be removed from Σ so that the remaining set is a regular surface?

Since it is not clear by the description, I will assume this surface is a double sided "cone" and extends to infinity in both directions.

a)

solution:

We can achieve a two dimensional parametrization whose trace is $\sum$, by parametrizing C by $u \in (a, b)$ and the lines through O and points on C by $v \in (-\infty, \infty)$

The parametrized surface, whose trace is $\sum$, is defined as

$$x : (a, b) \times \mathbb{R} \to \mathbb{R}^3 \quad \text{where} \quad x(u, v) = (v\alpha_x(u), v\alpha_y(u), v\alpha_z(u))$$

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b)

solution:

We have that

$$\frac{\partial x}{\partial u} = v\alpha'(u) \quad \text{and} \quad \frac{\partial x}{\partial v} = \alpha(u)$$

Which gives

$$\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v} = v(\alpha'(u) \wedge \alpha(u))$$

So $\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v} = 0$ when $v = 0$ or $\alpha'(u) \wedge \alpha(u) = 0$

So the critical points occur on the lines $\{(u, v) \in (a, b) \times \mathbb{R} \mid \alpha'(u) \wedge \alpha(u) = 0\}$ and on the u-axis.

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c)

solution:

To make $\sum$ a regular surface we should remove the image of the critical points. The image of the u-axis is the point $O = (0, 0, 0)$

The image of a line $\{(u, v) \in (a, b) \times \mathbb{R} \mid \alpha'(u) \wedge \alpha(u) = 0\}$ is a line through the origin and the point $\alpha(u)$

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g) Problem 15 on page 82, Section 2-3, Baby Do Carmo.

a) It was shown in the book that all parameterizations of a surface are diffeomorphic to one another and for any parameterizations $\alpha, \beta, \alpha^{-1} \circ \beta$ is diffeomorphic.
b)

$$
\begin{aligned}
|\int_{\tau_0}^{\tau} |\beta'(\tau)|d\tau| &= |\int_{\tau_0}^{\tau} |(\alpha \circ h)'(\tau)|d\tau| \\
&= |\int_{t_0}^{t} |(\alpha)'(t)|dt|
\end{aligned}
$$

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