Differential Geometry
Homework 7
Monday, November 11 2019

A.a) Write up a proof for the Key Theorem on page 216, Baby Do Carmo.

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B.a) Carry out the details for Example 5, page 162, Baby Do Carmo. (Including the application to a geometric interpretation of the Dupin indicatrix, that is from page 164 to 165, Baby Do Carmo.)

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C.a) Problem 2 on page 151, Section 3-2, Baby Do Carmo.

Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar

Let S be the surface, let $N(p)$ be the orientation at a point $p \in T_p(S)$. Let the curve be $\alpha(t)$. This curve is in the intersection of our surface and our plane. For any point $\alpha(t_0)$ on the curve the tangent plane is always the same. Therefore $dN(\alpha(t_0))_{\alpha'(t_0)} = \vec{0}$. Then let $a = \frac{\alpha''(t_0)}{||\alpha''(t_0)||}$. So a is a unit vector in $T_{\alpha(t_0)}(S)$ perpindicular to $\alpha'(t_0)$. Then let $b = dN(\alpha(t_0))_a$. Then gaussian curvature is $det(dN(\alpha(t_0))) = 0$. Therefore, the curve is either parabolic or planar. ∎

C.b) Problem 6 on page 151, Section 3-2, Baby Do Carmo.

Show that the sum of the normal curvatures for any pair of orthogonal directions, at a point $p \in S$, is constant.

Let $\mathbf{x}_u, \mathbf{x}_v \in T_p(S)$ be an orthonormal basis for $T_p(S)$. Let $a, b = a_1\mathbf{x}_u + a_2\mathbf{x}_v, b_1\mathbf{x}_u + b_2\mathbf{x}_v \in T_p(S)$ be a pair of orthogonal vectors. The sum of their normal curvatures is,

$$\begin{aligned}
- < dN_p(a), a > - < dN_p(b), b > &= - < a_1 dN_p(\mathbf{x}_u) + a_2 dN_p(\mathbf{x}_v), a_1\mathbf{x}_u + a_2\mathbf{x}_v > \\
&\quad - < b_1 dN_p(\mathbf{x}_u) + b_2 dN_p(\mathbf{x}_v), b_1\mathbf{x}_u + b_2\mathbf{x}_v > \\
&= -(a_1^2 < dN_p(\mathbf{x}_u), \mathbf{x}_u > + \\
&\quad a_1a_2 < dN_p(\mathbf{x}_u), \mathbf{x}_v > + \\
&\quad a_2a_1 < dN_p(\mathbf{x}_v), \mathbf{x}_u > + \\
&\quad a_2^2 < dN_p(\mathbf{x}_v), \mathbf{x}_v > + \\
&\quad b_1^2 < dN_p(\mathbf{x}_u), \mathbf{x}_u > + \\
&\quad b_1b_2 < dN_p(\mathbf{x}_u), \mathbf{x}_v > + \\
&\quad b_2b_1 < dN_p(\mathbf{x}_v), \mathbf{x}_u > + \\
&\quad b_2^2 < dN_p(\mathbf{x}_v), \mathbf{x}_v >) \\
= -((a_1^2 + b_1^2) < dN_p(\mathbf{x}_u), \mathbf{x}_u > &+2(a_1a_2 + b_1b_2) < dN_p(\mathbf{x}_u), \mathbf{x}_v > +(a_2^2 + b_2^2) < dN_p(\mathbf{x}_v), \mathbf{x}_v >) \\
= -((a_1^2 + a_2^2) < dN_p(\mathbf{x}_u), \mathbf{x}_u > &+2(\cancel{a_1a_2 - a_2a_1}) < dN_p(\mathbf{x}_u), \mathbf{x}_v > +(a_2^2 + a_1^2) < dN_p(\mathbf{x}_v), \mathbf{x}_v >) \\
= -(< dN_p(\mathbf{x}_u), \mathbf{x}_u > &+ < dN_p(\mathbf{x}_v), \mathbf{x}_v >)
\end{aligned}$$

So the sum of the curvatures does not depend on the pair chosen. ∎

C.c) Problem 8 on page 151, Section 3-2, Baby Do Carmo.

Desribe the region of the unit sphere covered by the Gauss map of the following surfaces:

(a) Paraboloid of revolution $z = x^2 + y^2$

(b) Hyperboloid of revolution $x^2 + y^2 - z^2 = 1$

(c) Catenoid $x^2 + y^2 = cosh^2 z$

(a) The bottom hemisphere

(b) the entire sphere

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C.d) Problem 17 on page 152, Section 3-2, Baby Do Carmo.

Show that if $H \equiv 0$ on S and S has no planar points, then the Gauss map $N : S \to S^2$ has the following property

$$\langle dN_p(w_1), dN_p(w_2)\rangle = -K(p)\langle w_1, w_2\rangle, \text{ for all } p \in S \text{ and all } w_1, w_2 \in T_p(S)$$

Show that the above condition implies that the angle of two intersecting curves on S and the angle of their spherical images (cf. Exercise 9) are equal up to a sign.

For reference

Exercise 9: Prove that

(a) The image $N \circ \alpha$ by the Gauss map $N : S \to S^2$ of a parametrized regular curve $\alpha : I \to S$ which contains no planar or parabolic points is a parametrized regular curve on the sphere S^2 (called the *spherical image* of α).

(b) If α is a line of curvature, and k is its curvature at p, then

$$k = |k_n K_N|$$

where k_n is the normal curvature at p along the tangent line of C and k_N is the curvature of the spherical image $N(C) \subset S^2$ at $N(p)$.

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