
Differential Geometry
Homework 8
Monday, November 18 2019

C.a) Problem 5 on page 168, Section 3-3, Baby Do Carmo.

Consider the parametrized surface (Enneper's surface)

$$x(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right)$$

Show that

(a) The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, \quad F = 0$$

(b) The coefficients of the second fundamental form are

$$e = 2, \quad g = -2, \quad f = 0$$

(c) The principal curvatures are

$$k_1 = \frac{2}{(1 + u^2 + v^2)^2}, \quad k_2 = \frac{-2}{(1 + u^2 + v^2)^2}$$

Done compuationally. See https://weiqinggu.github.io/Math142/homework/hw8/curvature.pdf

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C.b) Problem 1 on page 185, Section 3-4, Baby Do Carmo.

Prove that the differentiability of a vector field does not depend on the choice of a coordinate system.

A change of coordinates is just a matrix multiplication. A matrix multiplication is multiplying by constants and then summing. Differentiable functions are closed under addition and multicplication so a differentiable vector field will still be differentialbe after a

change of coordinates.
Let M be a change of coordinates for a vector field $w(a,b) = (w_1(a,b), \cdots)$. Then for all i,

$$\nabla w_i(M(a,b)) = M\nabla w_i$$

So we still have a derivative everywhere. So it is still diferentiable. ■

C.c) Problem 2 on page 185, Section 3-4, Baby Do Carmo.

Prove that the vector field obtained on the torus by parametrizing all its meridians by arc length and taking their tangent vectors (Example 1) is differentiable.

The tangent vector for a curve paramterized by arclength is just the derivative of the curve. So the problem is reduced to showing that the merdians parameterized by arclength are twice differentiable. Reparameterizing a curve does not change that it is twice differentiable. So we can just take some arbitrary parameterization of a meridian like $\alpha(a,b) = \sqrt{a^2+b^2}$. This is twice differentiable. ■

C.d) Problem 5 on page 186, Section 3-4, Baby Do Carmo.

Let S be a surface and let $x : U \to S$ be a parametrization of S. If $ac - b^2 < 0$, show that

$$a(u,v)(u')^2 + 2b(u,v)u'v' + c(u,v)(v')^2 = 0$$

can be factored into two distinct equations, each of which determines a field of directions on $X(U) \subset S$. Prove that these two fields of directions are orthogonal if and only if

$$Ec - 2Fb + Ga = 0$$

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C.e) Problem 8 on page 187, Section 3-4, Baby Do Carmo.

Show that if w is a differentiable vector field on a surface S and $w(p) \neq 0$ for some $p \in S$, then it is possible to parametrize a neighborhood of p by $x(u,v)$ in such a way that $x_u = w$.

Let $M : R^2 \to R^3$ be a linear transformation. Let $y = x(M(u,v))$. Then $y_u = Mx_u$. Since $w \neq 0$ and $x_u \neq 0$ there exists some M such that $Mx_u = w$. So by choosing the right M we can always get a parameterization of S where $y_u = w$. ■

Extra Credit Problems

D.a) Problem 10 on page 187, Section 3-4, Baby Do Carmo.

Let T be the Torus of Example 6 of Sec. 2-2 and define a map $\phi : \mathbb{R}^2 \to T$ by

$$\phi(u, v) = ((r\cos u + a)\cos v, (r\cos u + a)\sin v, r\sin u),$$

where u and v are the Cartesian coordinates of $\mathbb{R}^2$. Let $u = at, v = bt$ be a straight line in $\mathbb{R}^2$, passing by $(0, 0) \in \mathbb{R}^2$, and consider the curve in T $\alpha(t) = \phi(at, bt)$. Prove that

(a) ϕ is a local diffeomorphism.

(b) The curve $\alpha(t)$ is a regular curve; $\alpha(t)$ is a closed curve if and only if b/a is a rational number.

(c) If b/a is irrational, the curve $\alpha(t)$ is dense in T; that is, in each neighborhood of a point $p \in T$ there exists a point of $\alpha(t)$

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D.b) Problem 11 on page 187, Section 3-4, Baby Do Carmo.

Use the local uniqueness of trajectories of a vector field w in $U \subset S$ to prove the following result. Given $p \in U$, there exists a unique trajectory $\alpha : I \to U$ of w, with $\alpha(0) = p$, which is maximal in the following sense: Any other trajectory $\beta : J \to U$, with $\beta(0) = p$, is the restriction of α to J (i.e $J \subset I$ and $\alpha|_J = \beta$)

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D.c) Problem 12 on page 187, Section 3-4, Baby Do Carmo.

Prove that if w is a differentiable vector field on a compact surface S and $\alpha(t)$ is the maximal trajectory of w with $\alpha(0) = p \in S$, then $\alpha(t)$ is defined for all $t \in \mathbb{R}$.

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