Linear Algebra Review

Due date: ______________

1. For which of the following matrices are you *guaranteed* a real diagonal form or no real diagonal form at all without first determining the existence of an eigenbasis? Why?

$$A = \begin{pmatrix} 5 & 0 & -1 \\ 0 & 3 & 3 \\ -1 & 3 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 5 & 3 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad C = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad E = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad F = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$$

Solution. (a) We are guaranteed a real diagonal form without first determining the existence of an eigenbasis because A is symmetric.

(b) We are not guaranteed a real diagonal form. Since the eigenvalue 2 has multiplicity 2, we must determine wither there are 2 linearly independent eigenvectors for the eigenspace V_2.

(c) This matrix represents a rotation by 30°, so we know that there is *no* real diagonal form.

(d) This matrix represents a rotation by 90°, so we know that there is *no* real diagonal form.

(e) We know that there is *no* real diagonal form because E represents the shear

$$(x, y) \mapsto (x + ay, y).$$

That is, only the x-axis is invariant; everything else has been moved left or right.

(f) We find all the eigenvalues of F:

$$0 = \det(F - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1).$$

Hence, the eigenvalues are 4 and -1. Since these are distinct, we are guaranteed a real diagonal form.

2. Let

$$A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{pmatrix}.$$

(please turn over)

(a) Find the eigenvalues and corresponding eigenvectors of A.

Solution. We first find the characteristic equation of A:

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & -4 \\ 0 & 5-\lambda & 4 \\ -4 & 4 & 3-\lambda \end{vmatrix} = (3-\lambda)(\lambda-9)(\lambda+3).$$

Thus, the eigenvalues are $\lambda_1 = 3, \lambda_2 = 9, \lambda_3 = -3$. Since all the eigenvalues are distinct, A can be diagonalized.
For $\lambda_1 = 3$, we have

$$\begin{pmatrix} 1-3 & 0 & -4 \\ 0 & 5-3 & 4 \\ -4 & 4 & 3-3 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & 0 & -4 \\ 0 & 2 & 4 \\ -4 & 4 & 0 \end{pmatrix} \xrightarrow[-2R_2+R_3]{-2R_1+R_3} \begin{pmatrix} -2 & 0 & -4 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow v_1 = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}.$$

Similarly, for $\lambda_2 = 9$ we have $v_2 = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$ and for $\lambda_3 = -3$ we have $v_3 = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$.

(b) Is A similar to a diagonal matrix? If so, find a nonsingular matrix P such that $P^{-1}AP$ is diagonal. Is P unique? Explain.

Solution. Since A is symmetric, it can be diagonalized. Let

$$P = (v_1, v_2, v_3) = \begin{pmatrix} -2 & 1 & -2 \\ -2 & -2 & 1 \\ 1 & -2 & -2 \end{pmatrix}.$$

Then

$$P^{-1}AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -3 \end{pmatrix} = D.$$

However, P is not unique, since the eigenvectors associated to an eigenvalue are not unique.

(c) Find the eigenvalues of A^{-1}.

(please turn over)

Solution. Recall that, if A is invertible and has eigenvalue $\lambda \neq 0$, then $1/\lambda$ is an eigenvalue of A^{-1}. In our case, A is invertible since $\det A = \det PDP^{-1} = \det D = -81 \neq 0$ (and all the eigenvalues are nonzero). Thus, A^{-1} has eigenvalues $1/3, 1/9, -1/3$.

(d) Find the eigenvalues and corresponding eigenvectors of A^2.

Solution. If λ is a nonzero eigenvalue of A with associated eigenvector ξ, then

$$A^2\xi = A(\lambda\xi) = \lambda A\xi = \lambda^2\xi.$$

Hence, λ^2 is an eigenvalue of A^2 with associated eigenvector ξ. Hence, the eigenvalues of A^2 are $9, 81$, and 9, and their associated eigenvectors are $v_1 = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$, and $v_3 = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$.

3. Let $L : \mathcal{P}_2 \to \mathcal{P}_2$ be defined by

$$L(a + bt + ct^2) = (2a - c) + (a + b - c)t + ct^2.$$

(a) Find the matrix A representing L with respect to the standard basis of $\mathcal{P}_2$.

Solution. We note that $L(1) = 2 + t + 0t^2$, $L(t) = 0 + t + 0t^2$, and $L(t^2) = -1 + (-1)t + t^2$. Hence,

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Find all the eigenvalues of A. For each eigenvalue, find all eigenvectors associated with that eigenvalue.

Solution. Expanding along the bottom row, we compute

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & -1 \\ 1 & 1-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(1-\lambda).$$

Hence, the eigenvalues are $\lambda_1 = 1$ (multiplicity 2) and $\lambda_2 = 2$.

(please turn over)

Substituting in $\lambda_1 = 1$ and row-reducing, we find

$$\begin{pmatrix} 2-1 & 0 & -1 \\ 1 & 1-1 & -1 \\ 0 & 0 & 1-1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This system has solution $x = z$, $y = s =$ free, and $z = t =$ free, or

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ s \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Hence, eigenvalues associated with $\lambda_1 = 1$ are spanned by $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. For $\lambda_2 = 2$, we may similarly compute the eigenvector $v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(c) Find a matrix P such that $P^{-1}AP$ is diagonal.

Solution. Let

$$P = [v_1\, v_2\, v_3] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D.$$

(d) Find A^n where n is an integer. What is L^{100}?

Solution. Since $A = PDP^{-1}$, we have

$$\begin{aligned} A^n &= (PDP^{-1})^n \\ &= PD^nP^{-1} \\ &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1^n & 0 & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2^n & 0 & 1-2^n \\ 2^n-1 & 1 & 1-2^n \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence,

$$L^{100}(a + bt + ct^2) = \Big(a2^{100} + c(1-2^{100})\Big) + \Big(a(2^{100}-1) + b + c(1-2^{100})\Big)t + ct^2.$$

(please turn over)

4. Let A be an $n \times n$ real matrix.

(a) Prove that the coefficient of λ^{n-1} in the characteristic polynomial of A is given by $-\operatorname{tr} A$.

Solution. Expanding along the first row of $\lambda I_n - A$, we see that the characteristic polynomial $p_A(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n$ of A is given by

$$\begin{aligned} \det(\lambda I_n - A) &= \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix} \\ &= (\lambda - a_{11})C_{11} + (-a_{12})C_{12} + \cdots + (-a_{1n})C_{1n}, \end{aligned}$$

where C_{ij} is the i, j cofactor of $\lambda I_n - A$. Now, the expression involving λ^{n-1} in the characteristic equation must arise from the first term in this sum, since every other cofactor will contain only $n-2$ factors of λ. Applying the same argument to our computation of

$$C_{11} = \begin{vmatrix} \lambda - a_{22} & -a_{23} & \cdots & -a_{2n} \\ -a_{32} & \lambda - a_{33} & \cdots & -a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_n n \end{vmatrix},$$

we see that that the expression involving λ^{n-1} in the characteristic polynomial must arise from the product

$$(\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}) = \lambda^n - (a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} + \cdots .$$

Thus, $a_1 = -(a_{11} + a_{22} + \cdots + a_{nn}) = -\operatorname{tr} A$.

(b) Prove that $\operatorname{tr} A$ is the sum of the eigenvalues of A.

Solution. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, then $\lambda - \lambda_i$ $(i = 1, 2, \ldots, n)$ are factors of the characteristic polynomial

$$\begin{aligned} \det(\lambda I_n - A) &= \lambda_n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n \\ &= (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) \\ &= \lambda^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n)\lambda^{n-1} + \cdots + (-1)^n \lambda_1\lambda_2 \cdots \lambda_n. \end{aligned} \tag{1}$$

Thus, $a_1 = -\lambda_1 - \lambda_2 - \cdots - \lambda_n = -\operatorname{tr} A$. Hence, the trace of A is the sum of the eigenvalues of A.

(please turn over)

(c) Prove that the constant coefficient of the characteristic polynomial of A is $\pm$ the product of the eigenvalues of A.

Solution. We observe that in Eq. 1 above, the constant term is $a_n = (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n$. Also note that if $\lambda = 0$, then we have

$$\begin{aligned}(-1)^n \det A &= \det(-A) \\ &= a_n \\ &= (-1)^n \det A \\ &= (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n.\end{aligned}$$

Hence, $\det A$ is the product of the eigenvalues.

5. Let A be a 5×5 matrix. Suppose A has distinct eigenvalues $-1, 1, -10, 5, 2$.

(a) What is $\det A$? What is $\operatorname{tr} A$?

Solution. From the previous problem, we know that $\det A = -1 \times 1 \times (-10) \times 5 \times 2 = 100$ and $\operatorname{tr} A = -1 + 1 + (-10) + 5 + 2 = -3$.

(b) If A and B are similar, what is $\det B$? Why?

Solution. Since A and B are similar, there is some invertible matrix P such that $B = P^{-1}AP$. Hence,

$$\det B = \det(P^{-1}AP) = (\det P)^{-1}(\det A)(\det P) = \det A.$$

Thus, $\det B = 100$.

(c) Do you expect that all eigenvectors of A are mutually orthogonal? Why?

Solution. No, we can only expect that all the eigenvectors of A are linearly independent. However, since A is not symmetric, we are not guaranteed that they eigenvectors are mutually orthogonal.

6. *This is an extra credit-type problem.* Let $p_1(\lambda)$ be the characteristic polynomial of A_{11} and $p_2(\lambda)$ the characteristic polynomial of A_{22}. What is the characteristic polynomial of each of the following partitioned matrices?

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} A_{11} & A_{21} \\ 0 & A_{22} \end{pmatrix}$$

(please turn over)

Solution. Let $p_A(\lambda)$ and $p_B(\lambda)$ be the respective characteristic polynomials of A and B. Then

$$\begin{aligned} p_A(\lambda) &= |\lambda I - A| \\ &= \begin{vmatrix} \lambda I - A_{11} & 0 \\ 0 & \lambda I - A_{22} \end{vmatrix} \\ &= |\lambda_i - A_{11}||\lambda I - A_{22}| \\ &= p_1(\lambda)p_2(\lambda), \end{aligned}$$

where $p_1(\lambda)$ is the characteristic polynomial of A_{11} and $p_2(\lambda)$ is the characteristic polynomial of A_{22}. Similarly, $p_B(\lambda) = p_1(\lambda)p_2(\lambda)$.

7. Prove key theorems

(a) Prove that similar matrices have the same eigenvalues.

Solution. Let $B = P^{-1}AP$. Then

$$\begin{aligned} \det(\lambda I - B) &= \det(\lambda I - P^{-1}AP) \\ &= \det(\lambda P^{-1}IP - P^{-1}AP) \\ &= \det(P^{-1}(\lambda i - A)P) \\ &= \det(P^{-1})\det(\lambda I - A)\det P \\ &= \det(\lambda I - A), \end{aligned}$$

since $\det P^{-1} = (\det P)^{-1}$. Thus, similar matrices have the same eigenvalues.

(b) Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of a matrix A with associated eigenvectors $x_1, x_2, \ldots, x_k$. Prove that $x_1, x_2, \ldots, x_k$ are linearly independent.

Solution.

Let $a_1, a_2, \ldots a_k \in \mathbb{R}$, and suppose

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k = 0. \tag{2}$$

We prove by induction on k that $a_i = 0$ for $i = 1, 2, \ldots, k$. If $k = 1$, there is nothing to prove since any nonzero vector is linearly independent. Now, suppose that the statement holds for $k-1$. Applying A to both sides of Eq. 2, we see that

$$\begin{aligned} 0 &= a_1Ax_1 + a_2Ax_2 + \cdots + a_kAx_k \\ &= a_1\lambda_1x_1 + a_2\lambda_2x_2 + \cdots + a_k\lambda_kx_k. \end{aligned} \tag{3}$$

(please turn over)

Similarly, multiplying both sides of Eq. 2 by λ_k, we see that

$$0 = a_1\lambda_k x_1 + a_2\lambda_k x_2 + \cdots + a_k\lambda_k x_k. \tag{4}$$

Then, subtracting Eq. 3 from Eq. 4, we have

$$0 = a_1(\lambda_k - \lambda_1)x_1 + a_2(\lambda_k - \lambda_2)x_2 + \cdots + a_{k-1}(\lambda_k - \lambda_{k-1})x_{k-1}.$$

By the induction hypothesis, $x_1, x_2, \ldots, x_{k-1}$ are linearly independent, so

$$a_i(\lambda_k - \lambda_i) = 0$$

for $i = 1, 2, \ldots, k-1$. However, the λ_i are distinct, so $\lambda_i \neq \lambda_k$ for $k \neq i$. Thus, $a_1 = a_2 = \cdots = a_{k-1} = 0$.

Substituting these values into Eq. 2, we find that $a_k x_k = 0$. Since $x_k \neq 0$, we must have $a_k = 0$. Thus, $a_1 = a_2 = \cdots = a_k = 0$, so $x_1, x_2, \ldots, x_k$ are linearly independent, as desired.

(c) Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation defined by $L(X) = AX$. Let $V_\lambda = \{\xi \in \mathbb{R}^n \mid L(\xi) = \lambda\xi\}$. Prove V_λ is a subspace of $\mathbb{R}^n$. (This subspace is called the eigenspace associated with λ.)

Solution. There are two ways to show that V_λ is a subspace of $\mathbb{R}^n$. First, we directly show that V_λ satisfies the definition of a subspace. Suppose $\xi, \eta \in V_\lambda$. Then

$$L(\xi + \eta) = L(\xi) + L(\eta) = \lambda\xi + \lambda\eta = \lambda(\xi + \eta),$$

so $\xi + \eta \in V_\lambda$. Similarly, if $c \in \mathbb{R}$, then

$$L(c\xi) = cL(\xi) = c\lambda\xi = \lambda(c\xi),$$

so $c\xi \in V_\lambda$. Hence, V_λ is a subspace of $\mathbb{R}^n$ by definition.

Alternatively, we note that

$$\begin{aligned} V_\lambda &= \{\xi \in \mathbb{R}^n \mid L(\xi) = \lambda\xi\} \\ &= \{\xi \in \mathbb{R}^n \mid A\xi = \lambda\xi\} \\ &= \{\xi \in \mathbb{R}^n \mid A\xi - \lambda\xi = 0\} \\ &= \{\xi \in \mathbb{R}^n \mid (A - \lambda I)\xi = 0\} \\ &= \operatorname{null}(A - \lambda I). \end{aligned}$$

Since the nullspace of an $n \times n$ matrix is a subspace of $\mathbb{R}^n$, it follows that V_λ is a subspace of $\mathbb{R}^n$.

(please turn over)

(d) Let λ be an eigenvalue of A with multiplicity r. Let $\dim V_\lambda = s$. Prove $s \leq r$. (That is, the dimension of the eigenspace associated with λ is at most the multiplicity of λ.)

Solution. Let $\{x_1, x_2, \ldots, x_s\}$ be a basis of V_λ. We extend it to a basis $\{x_1, x_2, \ldots, x_s, x_{s+1}, \ldots, x_n\}$ of $\mathbb{R}^n$. Then

$$\begin{aligned}
L(x_1) &= \lambda x_1, \\
L(x_2) &= \lambda x_2, \\
&\vdots \\
L(x_s) &= \lambda x_s, \\
L(x_{s+1}) &= a_{s+1,1}x_1 + \cdots + a_{s+1,s}x_s + a_{s+1,s+1}x_{s+1} + \cdots + a_{s+1,n}x_n \\
&\vdots \\
L(x_n) &= a_{n,1}x_1 + \cdots + a_{n,s}x_s + a_{n,s+1}x_{s+1} + \cdots a_{n,n}x_n.
\end{aligned}$$

Thus, the matrix representation of L associated with the basis $\{x_1, x_2, \ldots, x_n\}$ is

$$M = \left(\begin{array}{cccc|cccc}
\lambda & 0 & \cdots & 0 & a_{s+1,1} & a_{s+2,1} & \cdots & a_{n,1} \\
0 & \lambda & \cdots & 0 & a_{s+1,2} & a_{s+2,2} & \cdots & a_{n,2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda & a_{s+1,s} & a_{s+2,s} & \cdots & a_{n,s} \\
\hline
0 & \cdots & \cdots & 0 & a_{s+1,s+1} & \cdots & \cdots & a_{n,s+1} \\
\vdots & \ddots & & \vdots & \vdots & \ddots & & \vdots \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & a_{s+1,n} & \cdots & \cdots & a_{n,n}
\end{array}\right) = \begin{pmatrix} \lambda I_s & A_1 \\ 0 & A_2 \end{pmatrix}.$$

Note that M and A are similar, since they represent the same linear transformation. Hence, the characteristic polynomial of L is

$$\begin{aligned}
f(x) &= \det(xI - A) \\
&= \det(xI - M) \\
&= \begin{vmatrix} (x-\lambda)I_s & -A_1 \\ 0 & xI_{n-s} - A_2 \end{vmatrix} \\
&= (x-\lambda)^s \det(xI_{n-s} - A_2) \\
&= (x-\lambda)^s g(x).
\end{aligned}$$

Since $g(x)$ might contain a factor of $(x - \lambda)$, it follows that the multiplicity r of λ is greater than or equal to s. Hence, $s \leq r$.

8. Inner products space: Let

$$u = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad v = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad w = \begin{pmatrix} 2 \\ 2 \\ -\sqrt{6} \end{pmatrix}.$$

(please turn over)

(a) Find $\|u\|, \|v\|$. Find a unit vector in the direction of u.

Solution. We compute $\|u\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$ and $\|v\| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$. Hence, the desired unit vector is

$$\hat{u} = \frac{u}{\|u\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

(b) Find the distance between v and w.

Solution. The distance between v and w is

$$\|v - w\| = \sqrt{(1-2)^2 + (-1-2)^2 + (0-(-\sqrt{6}))^2} = \sqrt{1+9+6} = 4.$$

(c) Find angle between u and v.

Solution. Let θ denote the desired angle, and recall that $u \cdot v = \|u\|\|v\| \cos\theta$. Hence,

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}.$$

By convention, we take $0 \le \theta \le \pi$, so $\theta = \arccos(1/2) = \pi/3$.

(d) Show that v and w are orthogonal.

Solution. Since $v \cdot w = 1 \times 2 + (-1) \times 2 + 0 \times (-\sqrt{6}) = 2 - 2 = 0$, v and w are orthogonal.

9. Useful facts for analysis

(a) Prove the Cauchy-Schwarz Inequality: If u and v are any vectors in an inner product space V, then $\langle u, v\rangle^2 \le \|u\|^2\|v\|^2$.

Solution. Let u and v be vectors in an inner product space V. Since the proof is trivial if $v = 0$, we may assume that $v \neq 0$. Consider the orthogonal projection

$$z = u - \frac{\langle u, v\rangle}{\langle v, v\rangle} v.$$

(please turn over)

Since z and v are orthogonal, we may apply the Pythagorean Theorem to

$$u = z + \frac{\langle u, v\rangle}{\langle v, v\rangle} v$$

to find

$$\|u\|^2 = \|z\|^2 + \frac{\langle u, v\rangle^2}{\langle v, v\rangle^2}\|v\|^2 = \|z\|^2 + \frac{\langle u, v\rangle^2}{\|v\|^2} \geq \frac{\langle u, v\rangle^2}{\|v\|^2}.$$

Hence, $\langle u, v\rangle^2 \leq \|u\|^2\|v\|^2$.

(b) Consider $\mathbb{R}^n$ with the standard inner product. Let $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$. Prove that

$$\left(\sum_{i=1}^n u_i v_i\right)^2 \leq \left(\sum_{i=1}^n u_i^2\right)\left(\sum_{i=1}^n v_i^2\right).$$

Solution. Note that $(\sum_{i=1}^n u_i v_i)^2 = \langle u, v\rangle$, $\sum_{i=1}^n u_i^2 = \langle u, u\rangle = \|u\|^2$, and $\sum_{i=1}^n v_i^2 = \langle v, v\rangle = \|v\|^2$. Thus, by the Cauchy-Schwartz Inequality, the inequality holds.

(c) Let V be the vector space of all continuous real-valued functions on the unit interval $[0, 1]$ with inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$. Prove

$$\left(\int_0^1 f(t)g(t)\,dt\right)^2 \leq \left(\int_0^1 f^2(t)\,dt\right)\left(\int_0^1 g^2(t)\,dt\right).$$

Solution. Note that $\left(\int_0^1 f(t)g(t)dt\right)^2 = \langle u, v\rangle$, $\int_0^1 f^2(t)dt = \langle u, u\rangle = \|u\|^2$, and $\int_0^1 g^2(t)dt = \langle v, v\rangle = \|v\|^2$. Thus, by the Cauchy-Schwartz Inequality, the inequality holds.

10. Positive definiteness: Let $C = [c_{ij}]$ be an $n \times n$ symmetric matrix and let V be an n-dimensional vector space with ordered basis $S = \{u_1, u_2, \ldots, u_n\}$. For $v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$ and $w = b_1u_1 + b_2u_2 + \cdots + b_nu_n$ in V, define

$$(v, w) = \sum_{i=1}^n \sum_{j=1}^n a_i c_{ij} b_j.$$

Prove that this defines an inner product on V if and only if C is a positive-definite matrix.

(please turn over)

Solution. Let $\mathbf{x} = [v]_S$ and $\mathbf{y} = [w]_S$ be the coordinate (column) vectors of v and w, respectively, with respect to the basis S. We note that

$$(v, w) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i c_{ij} b_j = \mathbf{x}^T C \mathbf{y}.$$

$\underline{\Rightarrow}$

Suppose that (v, w) defines an inner product on V. Then $(v, v) \geq 0$ and vanishes if and only if $v = 0$, so $\mathbf{x}^T C \mathbf{x} \geq 0$ if $\mathbf{x} \neq 0$ and $\mathbf{x}^T C \mathbf{x} = 0$ only if $x = 0$. Hence, C is positive-definite by definition.

$\underline{\Leftarrow}$

Conversely, suppose C is positive-definite. First, it follows by definition that $(v, v) = \mathbf{x}^T C \mathbf{x} \geq 0$, and vanishes if and only if $\mathbf{x} = 0$. Second, since C is symmetric,

$$(v, w) \quad = \mathbf{x}^T C \mathbf{y} = \langle \mathbf{x}, C\mathbf{y} \rangle = \langle C\mathbf{x}, y \rangle = \langle \mathbf{y}, C\mathbf{x} \rangle = \mathbf{y}^T C \mathbf{x} = (w, v),$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$. Third, if $\mathbf{z} = [u]_S$, then

$$\begin{aligned} (u + v, w) \quad &= [u + v]_S^T C [w]_S \\ &= (\mathbf{z}^T + \mathbf{x}^T) C \mathbf{y} \\ &= \mathbf{z}^T C \mathbf{y} + \mathbf{x}^T C \mathbf{y} \\ &= (u, w) + (v, w). \end{aligned}$$

Finally, if $r \in \mathbb{R}$, then

$$(rv, w) = [rv]_S^T C \mathbf{y} = r \mathbf{x}^T C \mathbf{y} = r(v, w).$$

Thus, if C is positive-definite, then $(v, w) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i c_{ij} b_j$ defines an inner product on V.

11. Let V be the vector space of all continuous functions on the interval $[-\pi, \pi]$. For f and g in V, define $\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) g(t)\, dt$.

(a) Show that this defines an inner product on V.

Solution. Let $f, g, h \in V$ and $r \in \mathbb{R}$. First, note that $\langle f, f \rangle = \int_{-\pi}^{\pi} f^2(t)\, dt \geq$ since $f^2(t) \geq 0$, and vanishes if and only if $f(t) = 0$. Second, since multiplication of functions is commutative,

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) g(t)\, dt = \int_{-\pi}^{\pi} g(t) f(t) = \langle g, f \rangle.$$

(please turn over)

Third, by the distributive law for functions, we have

$$\begin{aligned} \langle f+g, h\rangle &= \int_{-\pi}^{\pi}(f+g)h\,dt \\ &= \int_{-\pi}^{\pi}(fh+gh)\,dt \\ &= \int_{-\pi}^{\pi} fh\,dt + \int_{-\pi}^{\pi} gh\,dt \\ &= \langle f,h\rangle + \langle g,h\rangle. \end{aligned}$$

Finally, we have $\langle rf, g\rangle = \int_{-\pi}^{\pi} rfg\,dt = r\int_{-\pi}^{\pi} = r\langle f,g\rangle$. Hence, this defines an inner product on V.

(b) Show that the following set is an orthogonal set:

$$\{1, \cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos nt, \sin nt, \ldots\}.$$

Solution. First, we note that for any positive integer n, $\int_{-\pi}^{\pi}\cos nt\,dt = \int_{-\pi}^{\pi}\sin nt\,dt = 0$. Hence, 1 is orthogonal to every other element of the set.

Next, we note that $\cos mt\cos nt = \cos(m+n)t + \sin mt\sin nt$, so

$$\begin{aligned} \int_{-\pi}^{\pi}\cos mt\cos nt\,dt &= \cancelto{0}{\int_{-\pi}^{\pi}\cos(m+n)t\,dt} + \int_{-\pi}^{\pi}\sin mt\sin nt\,dt \\ &= \int_{-\pi}^{\pi}\sin mt\sin nt\,dt. \end{aligned}$$

However, using integration by parts, we find that

$$\begin{aligned} \int_{-\pi}^{\pi}\cos mt\cos nt\,dt &= \cancelto{0}{\left[\tfrac{1}{n}\cos mt\sin nt\right]_{-\pi}^{\pi}} + \tfrac{m}{n}\int_{-\pi}^{\pi}\sin mt\sin nt\,dt \\ &= \tfrac{m}{n}\int_{-\pi}^{\pi}\sin mt\sin nt\,dt. \end{aligned}$$

For $m \neq n$, this implies that $\int_{-\pi}^{\pi}\sin mt\sin nt\,dt = 0$, and therefore $\int_{-\pi}^{\pi}\cos mt\cos nt\,dt = 0$. Thus, each cosine element is orthogonal to every other cosine element, and each sine element is orthogonal to every other sine element in this set.

Finally, consider $\int_{-\pi}^{\pi}\sin nt\cos mt\,dt$. Since $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, we see that

$$\begin{aligned} \int_{-\pi}^{\pi}\sin nt\cos mt\,dt &= \cancelto{0}{\int_{-\pi}^{\pi}\sin(n+m)t\,dt} - \int_{-\pi}^{\pi}\cos nt\sin mt\,dt \\ &= -\int_{-\pi}^{\pi}\cos nt\sin mt\,dt. \end{aligned}$$

On the other hand, $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$, so

$$\begin{aligned} \int_{-\pi}^{\pi}\sin nt\cos mt\,dt &= \cancelto{0}{\int_{-\pi}^{\pi}\sin(n-m)t\,dt} + \int_{-\pi}^{\pi}\cos nt\sin mt\,dt \\ &= \int_{-\pi}^{\pi}\cos nt\sin mt\,dt. \end{aligned}$$

Hence, $\int_{-\pi}^{\pi}\cos nt\sin mt\,dt = 0$ for arbitrary n, m. Thus, all elements in this set are mutually orthogonal.

(please turn over)

(c) Convert the above set into an orthonormal set.

Solution. Since $\{1, \cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos nt, \sin nt, \ldots\}$ is an orthogonal set, we may obtain an orthonormal set by dividing each vector by its norm. We first compute

$$\begin{aligned}
\|1\|^2 &= \int_{-\pi}^{\pi} 1\, dt = 2\pi \\
\|\cos nt\|^2 &= \int_{-\pi}^{\pi} \cos^2 nt\, dt = \int_{-\pi}^{\pi} \tfrac{1+\cos 2nt}{2}\, dt \\
&= \tfrac{1}{2}\left[\cancelto{0}{\int_{-\pi}^{\pi} \cos 2nt\, dt} + \cancelto{2\pi}{\int_{-\pi}^{\pi} dt}\right] \\
&= \pi \\
\|\sin nt\|^2 &= \int_{-\pi}^{\pi} \sin^2 nt\, dt = \int_{-\pi}^{\pi} \tfrac{1-\cos 2nt}{2}\, dt = \pi.
\end{aligned}$$

Thus, we get an orthonormal basis

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos t}{\sqrt{\pi}}, \frac{\sin t}{\sqrt{\pi}}, \frac{\cos 2t}{\sqrt{\pi}}, \frac{\sin 2t}{\sqrt{\pi}}, \ldots, \frac{\cos nt}{\sqrt{\pi}}, \frac{\sin nt}{\sqrt{\pi}}, \ldots\right\}.$$

12. A linear transformation $L : V \to V$, where V is an n-dimensional Euclidean space, is called *orthogonal* if $\langle Lv, Lw\rangle = \langle v, w\rangle$.

(a) Let A be an $n \times n$ matrix. Show that A is orthogonal if and only if the columns (and rows) of A form an orthonormal basis for $\mathbb{R}^n$.

Solution. Since A is real, then $A^* = A^T$, where A^* is the adjoint of A. Note that $\langle Av, Aw\rangle = \langle v, A^T Aw\rangle$; hence, A is orthogonal if and only if $A^T A = I$.
We may write A as $(v_1, v_2, \ldots, v_n)$, where v_i is the ith column of A. Then

$$A^T A = \begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{pmatrix} \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} = \begin{pmatrix} v_1^T v_1 & v_1^T v_2 & \cdots & v_1^T v_n \\ v_2^T v_1 & v_2^T v_2 & \cdots & v_2^T v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n^T v_1 & v_n^T v_2 & \cdots & v_n^T v_n \end{pmatrix}.$$

Thus, $A^T A = I$ if and only if

$$v_i^T v_j = \langle v_i, v_j\rangle = \begin{cases} i = j & 1 \\ i \neq j & 0 \end{cases},$$

that is, if and only if $\{v_1, \ldots, v_n\}$ is an orthonormal basis for $\mathbb{R}^n$. The argument for the rows is identical with v_i and v_i^T interchanged.

(please turn over)

(b) Let S be an orthonormal basis for V and let the matrix A represent the orthogonal linear transformation L with respect to S. Prove that A is an orthogonal matrix.

Solution. Let $S = \{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis for V, and let $v_i = [Lu_i]_S$. Then

$$A = \begin{pmatrix} [Lu_1]_S & [Lu_2]_S & \cdots & [Lu_n]_S \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}.$$

Since L is orthogonal and S is orthonormal, we see that

$$v_i^T v_j = \langle v_i, v_j \rangle = \langle [Lu_i]_S, [Lu_j]_S \rangle = \langle [u_i]_S, [u_j]_S \rangle = \begin{cases} i = j & 1 \\ i \neq j & 0 \end{cases}.$$

Thus by part (a), A is an orthogonal matrix.

(c) Prove that for any vectors $u, v \in \mathbb{R}^n$, $\langle Lu, Lv \rangle = \langle u, v \rangle$ if and only if for any $u \in \mathbb{R}^n$, $\|Lu\| = \|u\|$.

Solution. Suppose $\langle Lu, Lv \rangle = \langle u, v \rangle$. Then if $v = u$, we see that

$$\|Lu\|^2 = \langle Lu, Lu \rangle = \langle u, u \rangle = \|u\|^2,$$

so $\|Lu\| = \|u\|$. Conversely, if $\|Lu\| = \|u\|$ for all u, then $\langle Lu, Lu \rangle = \langle u, u \rangle$. If $u = v + w$, then we may expand linearly to obtain

$$\langle Lv, Lv \rangle + \langle Lv, Lw \rangle + \langle Lw, Lv \rangle + \langle Lw, Lw \rangle = \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle. \quad (5)$$

We note that $\langle Lv, Lv \rangle = \langle v, v \rangle$ and $\langle Lw, Lw \rangle = \langle w, w \rangle$. Furthermore, since $v, w \in \mathbb{R}^n$, it follows that $\langle v, w \rangle = \langle w, v \rangle$. Thus, Eq. 5 becomes

$$2\langle Lv, Lw \rangle = 2\langle v, w \rangle,$$

so $\langle Lv, Lw \rangle = \langle v, w \rangle$ for all $v, w \in \mathbb{R}^n$.

(d) Let $L : V \to V$ be an orthogonal linear transformation. Show that if λ is an eigenvalue of L, then $|\lambda| = 1$.

Solution. Let λ be an eigenvalue of the orthogonal transformation $L : V \to V$. Since L is orthogonal, we know from part (c) that $\|Lv\| = \|v\|$ for all $v \in V$. Thus, for any nonzero eigenvector ξ associated with λ, we have

$$\|\xi\| = \|L\xi\| = \|\lambda\xi\| = |\lambda|\|\xi\|.$$

Since $\|\xi\| \neq 0$, it follows that $|\lambda| = 1$.

(please turn over)

13. Let W be the subspace of the Euclidean space $\mathbb{R}^4$ with standard inner product with basis $S = \{u_1, u_2, u_3\}$, where

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \qquad u_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Transform S to an orthonormal basis $T = \{w_1, w_2, w_3\}$ using the Gram-Schmidt process.

Solution. We define

$$w_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

We then compute

$$\begin{aligned} v_2 &= u_2 - \langle u_2, w_1 \rangle w_1 \\ &= \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + \left(\tfrac{2}{\sqrt{3}}\right) \tfrac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -1/3 \\ 2/3 \\ -1/3 \\ 1 \end{pmatrix}, \end{aligned}$$

and

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{5/3}} \begin{pmatrix} -1/3 \\ 2/3 \\ -1/3 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{15}} \begin{pmatrix} -1 \\ 2 \\ -1 \\ 3 \end{pmatrix}.$$

Finally, we compute

$$\begin{aligned} v_3 &= u_3 - \langle u_3, w_1 \rangle w_1 - \langle u_3, w_2 \rangle w_2 \\ &= \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + \left(\tfrac{1}{\sqrt{3}}\right) \tfrac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \left(\tfrac{2}{\sqrt{15}}\right) \tfrac{1}{\sqrt{15}} \begin{pmatrix} -1 \\ 2 \\ -1 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -4/5 \\ 3/5 \\ 1/5 \\ -3/5 \end{pmatrix} \end{aligned}$$

(please turn over)

and

$$w_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{35}} \begin{pmatrix} -4 \\ 3 \\ 1 \\ -3 \end{pmatrix}.$$

Thus, the desired orthonormal basis is

$$T = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{15}} \begin{pmatrix} -1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \frac{1}{\sqrt{35}} \begin{pmatrix} -4 \\ 3 \\ 1 \\ -3 \end{pmatrix} \right\}.$$

14. Orthogonal diagnilization of symmetric matrices.

(a) Let

$$A = \begin{pmatrix} -1 & 3 & 3 \\ 3 & -1 & 3 \\ 3 & 3 & -1 \end{pmatrix}.$$

Find a 3×3 matrix P with $P^{-1} = P^T$ such that $P^T A P = D$, where D is a 3×3 diagonal matrix.

Solution. We note that, since A is diagonal, we are guaranteed such a matrix P by the real spectral theorem; moreover, any matrix whose columns are eigenvectors of A and form an orthonormal basis of $\mathbb{R}^3$ will satisfy these conditions. To find P, therefore, we must first find all the eigenvalues of A. The characteristic equation is

$$\det(A - \lambda I) = (5 - \lambda)(4 + \lambda)^2 = 0,$$

so the eigenvalues are $\lambda_1 = -4$ (multiplicity 2) and $\lambda_2 = 5$. We find that the eigenspace associated with λ_1 has basis

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

which we may convert to the orthonormal basis

$$u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \qquad u_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

(please turn over)

using the Gram-Schmidt process. We also find the normal eigenvector associated with $\lambda_2 = 5$ is

$$u_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Thus, the desired matrix is

$$P = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix},$$

and

$$D = P^T A P = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

(b) (Extra credit) Show that all the eigenvalues of a real symmetric matrix are real numbers.

Solution. Let ξ be a nonzero eigenvector of a real symmetric matrix A with associated eigenvalue λ. Since A is real and symmetric, it is self-adjoint, and therefore $A^* = A$. Thus,

$$\lambda \|\xi\|^2 = \langle \lambda\xi, \xi \rangle = \langle A\xi, \xi \rangle = \langle \xi, A\xi \rangle = \langle \xi, \lambda\xi \rangle = \overline{\lambda} \|\xi\|^2.$$

Since $\|\xi\| \neq 0$, it follows that $\lambda = \overline{\lambda}$. Hence, λ is real.

(c) Show that if A is a symmetric real matrix, then eigenvectors that belong to distinct eigenvalues of A are orthogonal.

Solution. Suppose A is a real, symmetric matrix and let u, v be eigenvectors of A associated with the distinct eigenvalues λ and μ, respectively. By part (b), λ and μ are both real. We note that A is self-adjoint, and therefore

$$\lambda \langle u, v \rangle = \langle \lambda u, v \rangle = \langle Au, v \rangle = \langle u, Av \rangle = \langle u, \mu v \rangle = \mu \langle u, v \rangle.$$

Since $\lambda \neq \mu$ by assumption, it follows that $\langle u, v \rangle = 0$. Hence, u and v are orthogonal.

(d) Prove that a symmetric matrix A is positive-definite if and only if $A = P^T P$ for a nonsingular matrix P.

(please turn over)

Solution. Suppose A is a positive-definite matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Since A is positive-definite, there is an orthonormal matrix Q such that

$$A = Q \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} Q^T.$$

Since all the eigenvalues of a positive-definite matrix are positive, we may write

$$\begin{aligned} A &= Q \begin{pmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \end{pmatrix} \begin{pmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \end{pmatrix} Q^T \\ &= QDDQ^T. \end{aligned}$$

Let $P = DQ^T$. Then $P^T = (DQ^T)^T = QD^T = QD$. Hence, $A = P^T P$. Conversely, suppose $A = P^T P$. If $\mathbf{x} \in \mathbb{R}^n$, then

$$\mathbf{x}^T A \mathbf{x} = \mathbf{x} P^T P \mathbf{x} = (P\mathbf{x})^T (P\mathbf{x}) = \langle P\mathbf{x}, P\mathbf{x} \rangle = \|P\mathbf{x}\|^2.$$

Hence, $\mathbf{x}^T A \mathbf{x} > 0$ if $\mathbf{x} \neq 0$ and $\mathbf{x}^T A \mathbf{x} = 0$ if $\mathbf{x} = 0$. Thus, A is positive-definite.

(e) Prove that if the matrix A is similar to a diagonal matrix, then A is similar to A^T.

Solution. Since A is similar to a diagonal matrix, there is some invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$. Thus,

$$\begin{aligned} A^T &= (P^{-1})^T D^T P^T \\ &= (P^{-1})^T D P^T \\ &= (P^{-1})^T P^{-1} P D P^{-1} P P^T \\ &= (P^T)^{-1} P^{-1} A P P^T \\ &= (PP^T)^{-1} A (PP^T). \end{aligned}$$

Thus, for $Q = PP^T$, $A = Q^{-1} A Q$. Since P is invertible, so too is Q. Hence, A and A^T are similar.

15. Applications

(please turn over)

(a) (Applications to solving ODE systems): Consider two adjoining cells separated by a permeable membrane and suppose that a fluid flows from the first cell to the second one at a rate (in milliliters per minute) that is numerically equal to three times the volume (in milliliters) of the fluid in the first cell. It then flows out of the second cell at a rate (in milliliters per minute) that is numerically equal to twice the volume in the second cell. Let $x_1(t)$ and $x_2(t)$ denote the volumes of the fluid in the first and second cells at time t, respectively. Assume that initially the first cell has 40 milliliters of fluid, while the second one has 5 milliliters of fluid. Find the volume of fluid in each cell at time t.

Solution. The change in volume of the fluid in each cell is the difference between the amount flowing in and the amount flowing out. Since no fluid flows into the first cell, we have

$$x_1'(t) = -3x_1(t),$$

where the minus sign indicates that the fluid is flowing out of the cell. The flow $3x_1(t)$ from the first cell flows into the second cell. The flow out of the second cell is $2x_2(t)$. Thus, the change in volume of the fluid in the second cell is given by

$$x_2'(t) = 3x_1(t) - 2x_2(t).$$

We therefore have the linear system, expressed in matrix form,

$$\mathbf{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = A\mathbf{x}.$$

Since the matrix A is lower diagonal, we may read off the eigenvalues to be $\lambda_1 = -3$ and $\lambda_2 = -2$. We compute the associated eigenvectors to be

$$v_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, the general solution is given by

$$\begin{aligned} \mathbf{x}(t) &= b_1 v_1 e^{\lambda_1 t} + b_2 v_2 e^{\lambda_2 t} \\ &= b_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-3t} + b_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}. \end{aligned}$$

From the initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 40 \\ 5 \end{pmatrix},$$

we find that $b_1 = 40$ and $b_2 = 125$. Thus, the volume of fluid in each cell at time t is given by

$$\begin{aligned} x_1(t) &= 40e^{-3t} \\ x_2(t) &= -120e^{-3t} + 125e^{-2t}. \end{aligned}$$

(please turn over)

(b) (Markov chain) Consider a plant that can have red flowers (R), pink flowers (P), or white flowers (W), depending upon the genotypes RR, RW, and WW. When we cross each of these genotypes with a genotype RW, we obtain the transition matrix

$$M = \begin{pmatrix} 0.5 & 0.25 & 0.0 \\ 0.5 & 0.5 & 0.5 \\ 0.0 & 0.25 & 0.5 \end{pmatrix}.$$

Suppose that each successive generation is produced by crossing only with plants of RW genotype. When the process reaches equilibrium, what percentage of the plants will have red, pink, or white flowers?

Solution. Let $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ denote the equilibrium percentages of the plants, and note that $Mv = v$. Rearranging, we can find v by solving the matrix equation $M - I = \mathbf{0}$. This corresponds to the following system in three variables:

$$\begin{array}{rcrcrcl} -0.5x & + & 0.25y & & & = & 0 \\ 0.5x & - & 0.5y & + & 0.5z & = & 0 \\ & & 0.25y & - & 0.5z & = & 0 \end{array}$$

Solving this systems yields the eigenvector $v = \begin{pmatrix} x \\ 2x \\ x \end{pmatrix}$. Since the sum of the entries of v must equal 1, we conclude that $v = \begin{pmatrix} 0.25 \\ 0.5 \\ 0.25 \end{pmatrix}$. Thus, at equilibrium 1/4 of plants will have red flowers, 1/2 will have pink flowers, and 1/4 will have white flowers.

16. Applications to Fibonacci sequence

(a) Recall $x_n = x_{n-1} + x_{n-2}$.
To use linear algebra we define the following system of equations,

$$\begin{cases} x_n = x_{n-1} + y_{n-1} \\ y_n = x_{n-1} \end{cases} \tag{6}$$

We can rewrite this in matrix form,

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$

(please turn over)

Therefore,

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Compute the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Solution. The characteristic equation of A is

$$0 = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{vmatrix} = \lambda(\lambda - 1) - 1 = \lambda^2 - \lambda - 1 = (\lambda - \phi)(\lambda - \overline{\phi}),$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\overline{\phi} = \frac{1-\sqrt{5}}{2}$. To find the eigenvector associated with $\lambda_1 = \phi$, we compute

$$\begin{pmatrix} \phi - 1 & -1 \\ -1 & \phi \end{pmatrix} = \begin{pmatrix} -\frac{1-\sqrt{5}}{2} & -1 \\ -1 & \frac{1+\sqrt{5}}{2} \end{pmatrix} \xrightarrow{-\overline{\phi}R_2 + R_1} \begin{pmatrix} 0 & 0 \\ -1 & \phi \end{pmatrix};$$

hence, the eigenspace associated with $\lambda_1 = \phi$ is spanned by $v_1 = \begin{pmatrix} \phi \\ 1 \end{pmatrix}$. Similarly, we find that the eigenspace associated with $\lambda_2 = \overline{\phi}$ is spanned by $v_2 = \begin{pmatrix} \overline{\phi} \\ 1 \end{pmatrix}$.

(b) Verify that if $A = PBP^{-1}$ and k is a positive integer, then $A^k = PB^kP^{-1}$.

Solution. We prove this by induction on k. Suppose $A = PBP^{-1}$. If $k = 1$, then $A^k = A = PBP^{-1} = PB^1P^{-1}$, as desired. Now, assume the statement holds for $k = n - 1$. Then

$$A^n = A^{n-1}A = PB^{n-1}P^{-1}PBP^{-1} = PB^{n-1}BP^{-1} = PB^nP^{-1},$$

as desired. Hence, for any positive integer k, $A^k = PB^kP^{-1}$.

(c) Using a hand calculator or MATLAB, compute f_8, f_{12}, and f_{20}, where f_n is the nth Fibonacci number, starting with $f_0 = f_1 = 1$.

Solution. To compute these values, we may use Binet's formula,

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right],$$

(please turn over)

which is obtainable by using the eigenvalues and eigenvectors from part (a) to compute $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} 1 \\ 1 \end{pmatrix} = PD^kP^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus, we find

$$f_8 = 34, \qquad f_{12} = 233, \qquad f_{20} = 10,946.$$

17. Determine which of the given quadratic forms in three variables are equivalent:

$$\begin{aligned} g_1(\mathbf{x}) &= x_1^2 + x_2^2 + x_3^3 + 2x_1x_2 \\ g_2(\mathbf{x}) &= 2x_2^2 + 2x_3^2 + 2x_2x_3 \\ g_3(\mathbf{x}) &= 3x_2^2 - 3x_3^2 + 8x_2x_3 \\ g_4(\mathbf{x}) &= 3x_2^2 + 3x_3^2 - 4x_2x_3. \end{aligned}$$

Solution. We can determine which quadratic forms are equivalent by converting them to matrices and comparing signatures and ranks. Two quadratic forms are equivalent if and only if they have equal ranks and signatures.

$g_1(x) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2$

The matrix form of g_1 is $M_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The characteristic polynomial is

$$\begin{aligned} \det(\lambda I - M_1) &= (\lambda - 1)[(\lambda - 1)^2] + 1[-1(\lambda - 1)] + 0 \\ &= \lambda^3 - 3\lambda^2 + 2\lambda = \lambda(\lambda - 2)(\lambda - 1). \end{aligned}$$

Thus, M_1 has eigenvalues $0, 1,$ and 2 and signature $2-0 = 2$. It is clear that $\operatorname{rank}(M_1) = 2$.

$g_2(x) = 2x_2^2 + 2x_3^2 + 2x_2x_3$

The matrix form of g_2 is $M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. The characteristic polynomial is

$$\det(\lambda I - M_2) = \lambda[(\lambda - 2)^2 - 1] = \lambda(\lambda - 3)(\lambda - 1).$$

Thus, M_2 has eigenvalues $0, 1,$ and 3 and signature $2-0 = 2$. Since the two nonzero rows are not multiples of each other, they are linearly independent. Thus, $\operatorname{rank}(M_2) = 2$.

(please turn over)

$g_3(x) = 3x_2^2 - 3x_3^2 + 8x_2x_3$

The matrix of g_3 is $M_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$. Since the two nonzero rows are not multiples of each other, they are linearly independent and $\operatorname{rank}(M_3) = 2$. The characteristic polynomial is

$$\det(\lambda I - M_3) = \lambda[(\lambda - 3)^2 - 16] = \lambda(\lambda - 7)(\lambda + 1).$$

Thus, M_3 has eigenvalues $-1, 0,$ and 7 and signature $1 - 1 = 0$.

$g_4(x) = 3x_2^2 + 3x_3^2 - 4x_2x_3$

The matrix of g_4 is $M_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{pmatrix}$. Since the two nonzero rows are not multiples of each other, they are linearly independent and $\operatorname{rank}(M_3) = 2$. The characteristic polynomial is

$$\det(\lambda I - M_4) = \lambda[(\lambda - 3)^2 - 4] = \lambda(\lambda - 1)(\lambda - 5).$$

Thus, M_4 has eigenvalues $0, 1,$ and 5 and signature $2 - 0 = 2$.

Since g_1, g_2, and g_4 all have rank 2 and signature 2, they are equivalent.

18. Which of the following matrices are positive-definite?

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix}.$$

Solution. Recall from Theorem 6.12 that a symmetric matrix M is positive definite if and only if all the eigenvalues of M are positive. We can therefore determine whether $A, B,$ and E are positive definite by examining their eigenvectors.

The eigenvalues of A are $1, 1,$ and 4, so A is positive definite. The eigenvalues of B are $4 \pm \sqrt{5}$, so B is also positive definite. The eigenvalues of E are $3 \pm \sqrt{13}$, so E is not positive definite.

To determine whether C is positive-definite, we let $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be a nonzero vector and compute

$$\begin{aligned} v^T C v &= \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= x^2 + 4xy + 2y^2 + 5xz + 6yz + 3z^2. \end{aligned}$$

(please turn over)

From this, we determine that for $v = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $v^T C v = -1 < 0$. Thus, C is not positive definite.

19. Let $g(\mathbf{x}) = 3x_1^2 - 3x_2^2 - 3x_3^2 + 4x_2x_3$ be a quadratic form in three variables.

(a) Find a quadratic form in the type given in the Principal Axis Theorem that is equivalent to g. What is the rank of g? What is the signature of g?

Solution. We first write the matrix form of g, $M = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 2 & -3 \end{pmatrix}$. We can then compute the characteristic polynomial

$$\det(\lambda I - M) = (\lambda - 3)[(\lambda + 3)^2 - 4] = (\lambda - 3)(\lambda + 1)(\lambda + 5)$$

to find that the eigenvalues of M are $\lambda_1 = -5$, $\lambda_2 = -1$, and $\lambda_3 = 3$. Thus, the desired equivalent quadratic form is $h(\mathbf{y}) = 3y_1^2 - y_2^2 - 5y_3^2$. It is clear that $\operatorname{rank}(g) = \operatorname{rank}(M) = 3$, and we can compute the signature of g to be $1 - 2 = -1$.

(b) Identify the surface $g(\mathbf{x}) = 9$.

Solution. If we set $g(\mathbf{x}) = 9$, we obtain a hyperboloid of two sheets. This is evident from the fact that g has two negative eigenvalues and one positive eigenvalue.