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Differential Geometry
Notes
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ρ is differentable at p, if $y^{-1} \circ \rho \circ x$ is differentible and well defined. So this means compositions of differentible functions are differentible.

The differential map is a linear map. It maps $span([1,0,0],[0,1,0])$ to the tangent plane. We can use it to change into the local cordinates system.

Consider a vector space V of finite dimension. then having an inner product on V is equivalent to having a norm on V. $||v+w||^2 =< v+w, v+w >=< v,v > +2 < v,w > + < w,w > \implies < v,w >= ||v+w||^2 - ||v||^2 - ||w||^2$. We've already shown in a previous lecture that we can get a norm from an inner product. Now we've shown that we can get an inner product from a norm. So they're equivalent.

Now we put a measurement on a manifold. We call a manifold with a measurement on it a riemannian manifold. For a reegular surface, the measurement is just the norm squared.

First fundamental form

Anything that eats a vector and spits out a number is a form. We can use the first fundamental form to find lengths and angles on the manifold.

The **first fundamental form** of $\mathbb{R}^3 \supset S$ induces on each tangent plane $T_p(S)$ of a regualr surface S an inner product, to be denoted by $<,>_p$. If $w_1, w_2 \in T_p(S) \subset \mathbb{R}^3$, then $< w_1, w_2 >$ is equal to the innner product of w_1, w_2 as vectors in $\mathbb{R}^3$Since this inner product is a symmetric bilinear form therre is a corresponding quadratic form $I_p : T_p(S) \to \mathbb{R}$ given by,

$$I_p(w) =< w,w >_p= ||w||^2 \geq 0$$

The quadratic fom I_p on $T_p(S)$ defined above is called the firsst fundamental form of the regular surface S at $p \in S$.

Second fundamental form

Let S' be the set of curves tangent to some $v \in S$. k_1 is defined as the largeset curvature of any curve in S'. k_2 is defined as the smallest curvature of any curve in S'. Gaussian cuvature is defind as $k_1 k_2$. Meean curvature iss $\frac{k_1+k_2}{2}$

The second fundamental form is equivalent to gaussian curvature in $\mathbb{R}^3$.

How to find first fundmanet form in local coordinates

Given $w \in T_p(S)$, find a curve $\alpha(t)$ on S such that $\alpha(t) = x(u(t), v(t))$ with $\alpha'(t) =$

$x_u u'(t) + x_v v'(t)$, $\alpha'(0) = x_u u'(0) + x_v v'(0) = [x_u, x_v] \begin{bmatrix} u'(0) \\ v'(0) \end{bmatrix}$

Then,

$$\begin{aligned} I_p(w) &= ||w||^2 \\ &= < u'(0)x_u + v'(0)x_v, u'(0)x_u + v'(0)x_v > \\ &= u'(0)^2 < x_u, x_u > +2u'(0)v'(0) < x_u, x_v > +v'(0)^2 < x_v, x_v > \\ &= [u'(0), v'(0)] \begin{bmatrix} < x_u, x_u > & < x_u, x_v > \\ < x_u, x_v > & < x_v, x_v > \end{bmatrix} [u'(0), v'(0)] \end{aligned}$$

$E =< x_u, x_u >, F =< x_u, x_v >, G =< x_v, x_v >$ are called the first fundamental cooefi-cients.

We can generlize this to a manifold. Let n be the dimension of the manifold. Let p be a point on the manifold. Let $\{u_1, \cdots, u_{n-1}\}$ be a basis for the tangent space at p. Then the riemmani metric on the manifold is,

$$||w||^2_{riemmani} = w^T \begin{bmatrix} < x_{u_1}, x_{u_1} > & \cdots & < x_{u_1}, x_{u_{n-1}} > \\ & \ddots & \\ < x_{u_{n-1}}, x_{u_1} > & \cdots & < x_{u_{n-1}}, x_{u_{n-1}} > \end{bmatrix}$$

Angle at which two curves intersect can be calculated with,

$$\cos\theta = \frac{< \alpha'(t_0), \beta'(t_0) >}{||\alpha'(t_0)||||\beta'(t_0)||}$$

The area of a bounded region $R \in S$ on a regular surface S with parameterization x is,

$$\int\int_{x^{-1}(R)} ||x_u \times x_v|| dudv = \int\int_{x^{-1}(R)} \sqrt{EG - F} dudv$$

$x_u = (-r \sin u \cos v, -r \sin u \sin v, r \cos u)$
$x_v = (-(a + r \cos u) \sin v, (a + r \cos u) \cos v, 0)$

$$\begin{aligned} E &= r^2(\sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u) = r^2 \\ G &= (a + r \cos u)^2 \sin^2 v + (a + r \cos u)^2 \cos^2 v = (a + r \cos u)^2 \\ F &= r(a + r \cos u) \sin u \cos v \sin v - r(a + r \cos u) \sin u \sin v \cos v = 0 \end{aligned}$$

$$\begin{aligned} Area &= \int_{u=0}^{2\pi} \int_{v=0}^{2\pi} \sqrt{EG - F}\, du\, dv \\ &= \int\int \sqrt{r^2(a+r)^2 - 0}\, du\, dv \\ &= \int\int r(a+r)\, du\, dv \\ &= \int\int ra\, du\, dv + \int\int r^2 \cos u\, du\, dv \qquad = 2\pi(ra + (\sin 2\pi - \sin 0)) \\ &= 2\pi(ra + (0 - 0)) \\ &= 2\pi ra \end{aligned}$$

Why unit quaternion multiplicaiton represents a rotation in $\mathbb{R}^3$

Let H be $\mathbb{R}^4$. Let $q \in S^3 = \{q \in H : ||q|| = 1\}$. For each q dfeine a map $H \to H, x \mapsto q \times \bar{q}$. Note that $||q|| = 1\ implies ||q||^2 = 1 \implies q\bar{q} = 1 \implies q^{-1} = \bar{q}$
So $R_q(x) = qx\bar{q}$. We claim R_q is a rotation in $\mathbb{R}^3$.
To show R_q is a rotation we show R_q is linear and preserves the the norm. First we show R_q is a linear map. $R_q(cx + y) = q(cx + y)\bar{q} = cqx\bar{q} + qy\bar{q} = cR_q(x) + R_q(y)$.
Now we show that $||R_q(x)|| = ||x||$,

$$\begin{aligned} ||R_q(x)|| &= ||qx\bar{q}|| \\ &= ||q||||x||||\bar{q}|| \\ &= ||x|| \end{aligned}$$

On the homework we will find matrix representation of R_q. It is $[R_q(\vec{i}), R_q(\vec{j}), R_q(\vec{k})]$