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Differential Geometry
Notes
Monday, Nov 4th 2019

A big picture of geometry of gauss map

Motivation: We want to use maps and their differentials to study the surfaces. What kind of maps should we consider? *Gauss Map* $: S \to S^2, p \to N(p)$. Let p be a point on a surface S and let $\mathbf{x}_u, \mathbf{x}_v$ be a basis for $T_p(S)$. Then

$$N(p) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{||\mathbf{x}_u \times \mathbf{x}_v||}$$

The second fundamental form $II_p(v) = - < dN_p(v), v >$
2) We can diagonalize dN_p. Let k_1, k_2 be the eigenvalues. Recall that the eigenvalues are the principle curvatures. That means there exists an orthonormal basis $\{e_1.e_2\} \in T_p(S)$ such that,

$$dN_p(e_1) = k_1 e_1$$
$$dN_p(e_2) = k_2 e_2$$

$-k_1, -k_2$ are the max and min of $\{II_p(v) : v \in T_p(S)\}$. Each invariant characteristic of dN_p has geometric meaning.

(a) $II_p(v)i \triangleq$ normal curvature along v

(b) $k_1, k_2 \triangleq$ principal curvature

(c) $det(dN_p) \triangleq$ Gaussian curvature

(d) $-\frac{1}{2}\, tr(dN_p)$

The Gauss Map in local coordinates
Let S be a surface. Let $\mathbf{x}(u,v)$ be a paramaeterization at a point $p \in S$.
Then **local coordinates** are u, v. A neighborhood in local coordinates gets mapped to a neighborhood on the surface.
Let α be a regular curve. Then $dN_p(\alpha'(0)) = N_u u'(0) + N_v v'(0)$.
Note that $N_u, N_v \in T_p(S)$. Then $N_u = \begin{bmatrix} \mathbf{x}_u & \mathbf{x}_v \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then that matrix representation of dN_p in local coordinates is A.
The second fundamental form in local coordinates
Let S be a surface. Let $w = \alpha'(0)$ where α is a regular curve in S. Let $p = \alpha(0) \in S$. Then,

$$\begin{aligned} II_p(w) &= II_p(\alpha'(0)) \\ &\triangleq - < dN_p(w), w > \\ &= -u'(0)^2 < N_u, \mathbf{x}_u > - u'(0)v'(0)(< N_u, \mathbf{x}_v > + < N_v, \mathbf{x}_u >) - v'(0)^2 < N_v, \mathbf{x}_v > \end{aligned}$$

So $e = - < N_u, \mathbf{x}_u >, f = - < N_u, \mathbf{x}_v >, g = - < N_v, \mathbf{x}_v >$.
Calculating e, f, g is hard. Recall that $< N, \mathbf{x}_u >= 0$. Therefore, $< N_u, \mathbf{x}_u > + < N, \mathbf{x}_{uu} = 0$. Therefore, $- < N_u, \mathbf{x}_u >=< N, \mathbf{x}_{uu} >$. So then $e =< N, \mathbf{x}_{uu} >, f = - < N, \mathbf{x}_{vu} >, g =< N, \mathbf{x}_{vv} >$. So all you need is some derivatives, dot products, and cross products.
Claim

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = - \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}$$

Proof:
$-e =< N_u, \mathbf{x}_u >=< a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \mathbf{x}_u >= a_{11} < \mathbf{x}_u, \mathbf{x}_u > a_{21} < \mathbf{x}_u, \mathbf{x}_v >= a_{11}E + a_{21}F$.
Similarly, $-f = a_{11}F + a_{21}G, -g = a_{12}F + a_{22}G$. This matches the given equation.
The principal curvatures are the eigenvalues of dN_p so they are the roots of the polynomial $k^2 + k(a_{11} + a_{12}) + \det(dN_p)$.
$F = 0$ if and only if $\mathbf{x}_u \perp \mathbf{x}_v$.
If $F = 0$ then $k_1 = e/E, k_2 = g/G$. For any compact manifold at a point p there exists a parameterization such that $\mathbf{x}_u \perp \mathbf{x}_v$.

$$
\begin{aligned}
x(u,v) &= \begin{bmatrix} (a + r\cos u)\cos v \\ (a + r\cos u)\sin v \\ r\sin u \end{bmatrix} \\
\mathbf{x}_u &= \begin{bmatrix} -r\sin u\cos v \\ -r\sin u\sin v \\ r\cos u \end{bmatrix} \\
\mathbf{x}_v &= \begin{bmatrix} -(a + r\cos u)\sin v \\ (a + r\cos u)\cos v \\ 0 \end{bmatrix} \\
\mathbf{x}_u \times \mathbf{x}_v &= \begin{bmatrix} -r\cos u(a + r\cos u)\cos v \\ -r\cos u(a + r\cos u)\sin v \\ -r\sin u\cos^2 v(a + r\cos u) - r\sin u\sin^2 v(a + r\cos u) \end{bmatrix} \\
&= \begin{bmatrix} r\cos u(a + r\cos u)\cos v \\ -r\cos u(a + r\cos u)\sin v \\ -r\sin u(a + r\cos u) \end{bmatrix} \\
&= -r(a + r\cos u)\begin{bmatrix} \cos u\cos v \\ \cos u\sin v \\ \sin u \end{bmatrix} \\
N = \frac{\mathbf{x}_u \times \mathbf{x}_v}{||\mathbf{x}_u \times \mathbf{x}_v||} &= \begin{bmatrix} \cos u\cos v \\ \cos u\sin v \\ \sin u \end{bmatrix} \frac{1}{\cos^2 u\cos^2 v + \cos^2 u\sin^2 v + \sin^2 u} \\
&= \begin{bmatrix} \cos u\cos v \\ \cos u\sin v \\ \sin u \end{bmatrix} \\
\mathbf{x}_{uu} &= \begin{bmatrix} -r\cos u\cos v \\ -r\cos u\sin v \\ -r\sin u \end{bmatrix} \\
\mathbf{x}_{vu} &= \begin{bmatrix} r\sin u\sin v \\ -r\sin u\cos v \\ 0 \end{bmatrix} \\
\mathbf{x}_{vv} &= \begin{bmatrix} -(a + r\cos u)\cos v \\ -(a + r\cos u)\sin v \\ 0 \end{bmatrix} \\
e &= < N, \mathbf{x}_{uu} > \\
&= -r\cos^2 u\cos^2 v - r\cos^2 u\sin^2 v - r\sin^2 u \\
&= -r \\
f &= < N, \mathbf{x}_{vu} > \\
&= r\sin u\sin v\cos u\cos v - r\sin u\cos v\cos u\sin v \\
&= 0 \\
g &= < N, \mathbf{x}_{vv} > \\
&= -(a + r\cos u)(\cos^2 v\cos u + \sin^2 v\cos u) \\
&= -(a + r\cos u)\cos u \\
E &= < \mathbf{x}_u, \mathbf{x}_u > = r^2\sin^2 u\cos^2 v + r^2\sin^2 u\sin^2 v + r^2\cos^2 u
\end{aligned}
$$

$$
\begin{aligned}
&= 1 \\
F &=< \mathbf{x}_u, \mathbf{x}_v >= 0 \\
G &=< \mathbf{x}_v, \mathbf{x}_v >= (a + r\cos u)^2
\end{aligned}
$$

Then the curvature is,

$$\frac{(a + r\cos(u))^3 \cos(u)}{r}$$