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Differential Geometry
Notes
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Definition A diffeomorphism $\phi : S \to \bar{S}$ is an isometry if for all $p \in S$ and all pairs $w_1, w_2 \in T_p(S)$ we have,

$$< w_1, w_2 >_p = < d\phi_p(w_1), d\phi_p(w_2) >_{\phi(p)}$$

S and $\bar{S}$ are isometric.
The surface of a cylindar and the plane have the same first fundamental form. So they are isometric.

Let $\bar{u} = u, \bar{v} = \bar{a} \sinh v$.
Let $x(\bar{u}, \bar{v}) = (\bar{v} \cos \bar{u}, \bar{v} \sin \bar{u}, a\bar{v}) = (\bar{a} \sinh v \cos u, \bar{a} \sinh v \sin u, au)$.
Then $x_u = (-\bar{a} \sinh v \sin u, \bar{a} \sinh v \cos u, \bar{a})$,
$x_v = (\bar{a} \cosh v \cos u, \bar{a} \cosh v \sin u, 0)$.
Then $E = \bar{a}^2 \sinh^2 v + \bar{a}^2 = \bar{a}^2 \cosh^2 v, F = 0, G = \bar{a}^2 \cosh^2 v$.
Gauss Map Gauss Map is a map $S \to S^2$(a unit sphere), $p \mapsto N(p)$ where $N(p)$ is the unit normal vector to the surface at point p.
Self adjoint: A linear map $L : V^n \to V^n$ is self adjoint if $< L(v), w > = < v, L(w) >$. A matrix is self adjoint if and only if it is symetric.

(a) $ll_p(v) \triangleq$ Normal curvature along v

(b) $k_1, k_2 \triangleq$ Principal curvaure

(c) $\det dN_p \triangleq$ gaussian curvature

(d) $-\frac{1}{2} tr(dN_p) \triangleq$ mean curvature

Theorem: The differential dN_p is a map from the tangent plane on the sphere to the tangent plane on the surface. The planes are parallel. So $dN_p(x_u) = N_u, dN_p(x_v) = N_v$.
The normal vector is orthogonal to x_u, x_v. Therefore,

$$\begin{aligned} < N, x_u > &= 0 \\ < N, x_v > &= 0 \\ \implies < N_v, x_u > + < N, x_{uv} > &= 0 \\ < N_u, x_v > + < N, x_{vu} > &= 0 \\ \implies < N_u, x_v > &= < N_v, x_u > \\ \implies < dN_p(x_u), x_v > &= < x_u, dN_p(x_v) > \end{aligned}$$

Since dN_p and the inner product are linear we only have to show it holds true for basis vectors x_u, x_v.

The second fundamental form: The fact that $dN_p : T_p(S) \to T_p(S)$ is a self adjoint means we can associate to dN_p a quadratic form Q in $T_p(S)$ given by $Q(v) =< dN_p v, v >, v \in T_p(S)$. To obtain a geometic interpretation of the equadratic form, we need a few definitions. For rerasons that will soon become clear, we shall use the quadratic fomr Q.
Definition of second fundamental form: It is the quadratic form ll_p defined in $T_p(S)$ by $ll_p(v) = - < dN_p v, v >$.
Let C be a regular cuve in S passing through $p \in S$,. Let k be the curvature of C at p and let $\cos\theta =< N, n >$. $k_n = k\cos\theta$ is the normal curvature of C at p.
Pf of Euler's formula: Normal cuvature along a given direction in $T_p(S)$ can be calculated from the principal curvaturees, $k_n = k_1\cos^2\theta + k_2\sin^2\theta$.
$k_n = - < dN_p(v), v >= - < dN_p(\cos\theta e_1 + \sin\theta e_2), v >= - < \cos\theta dN_p(e_1) + \sin\theta dN_p(e_2), v >=< k_1\cos\theta e_1 + k_2\sin\theta e_2, \cos\theta e_1 + \sin\theta e_2 >= k_1\cos^2\theta + k_2\sin^2\theta$.
Gaussian curvature: $k_1 k_2$
Mean curvature: $\frac{k_1+k_2}{2}$
A point is elliptic if $\det dN_p > 0$, hyperbolic if $\det dN_p < 0$, parabolic if $\det dN_p = 0$ with $dN_p \neq 0$, and planar if $dN_p = 0$.