Differential Geometry Class Notes

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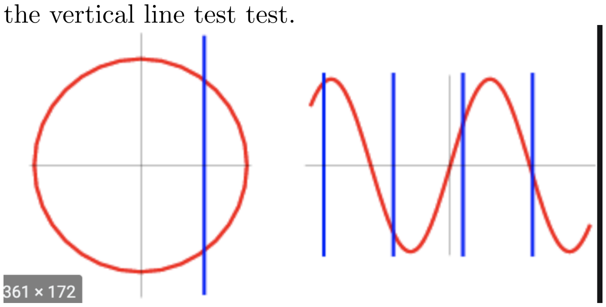
1 Regular Curves

Overview of key concepts will be covered in today's lecture. Definition A *parametrized differentiable curve* is a differentiable map $\alpha : I \to \mathbb{R}^n$ of an open interval $I = (a, b)$ of the real line $\mathbb{R}$ into $\mathbb{R}^n$.
Limitations of multivaraible functions for complex shapes: As we learned in 8th grade a function can return only one value for each input. This means that certain shapes are not possible. We see this visually with the vertical line test test.



But a circle can be modeled easily with a parametric curve. Use $I = (0, 2\pi), \alpha(t) = (cos(t), sin(t))$
Definition *Regular curves* are curves where at least the second derivative exists and $\alpha'(t) \neq 0$. These are curves to which you can fit data. we require $\alpha'(t) \neq 0$ to guarantee the existence of tangent vectors. Later we also require $\alpha''(t) \neq 0$ to guarantee the existence of normal vectors.

- $x^2 + y^2 = 1$
- $\alpha(t) = (\cos(t), \sin(t))$
- $\begin{cases} x = \sin(t) \\ y = \cos(t), 0 < t <= 2\pi \end{cases}$
- $\beta(t) = e^{it}$

Note: The second is a parametrized curve. Also the second and third have different orientations. The first one, does not tell us any orientations.

Let's see another example. A helix in $\mathbb{R}^3$. The parametrized helix can be written as a 3D curve

$$\gamma(t) = (\cos(t), \sin(t), t)$$

When the curve is parameterized by *arc length* (i.e., speed at each point is 1), then we can use the Frenet Frame to describe the curve locally. You can think of this as a three-arrowed axis which travels along the curve and describes the rate change of these frames along the curve.

Recall: a function being differentiable can be tested by looking at whether

$$\frac{\partial f_i}{\partial x_j}$$

exists and is continuous for all i and j (This is called C^1 differentiable). This is a sufficient condition for f being C^1 differentiable. However, when we speak of parameterized curves being "differentiable", we really mean smooth or infinitely differentiable, i.e. $f \in \mathbb{C}^\infty$

Recall: a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable only if each of its m component functions is differentiable.

Example Helix: $x^2 + y^2 = a^2$ in $\mathbb{R}^3$, such that z can be anything. Observe that this function, in $\mathbb{R}^2$, is a circle.

We may parameterize the circle as

$$\begin{cases} x(t) = a\cos(t) \\ y(t) = a\sin(t) \end{cases} \quad 0 < t <= 2\pi$$

which implies the parametrized curve being

$$\alpha(t) = (a\cos(t), a\sin(t))$$

The following is a different parametrized curve even they have the same trace.

$$\begin{cases} x = \sin(t) \\ y = \cos(t) \end{cases} \quad 0 < t <= 2\pi$$

which implies the parametrized curve is

$$\beta(t) = (a\sin(t), a\cos(t))$$

for the parameter $0 < t <= 2\pi$.

Consider another parametrized curve, which have same trace

$$\begin{cases} x = \cos(2\pi t) \\ y = \sin(2\pi t) \end{cases} \quad 0 < t <= 1$$

Which may be written as

$$\gamma(t) = (a\cos(2\pi t), a\sin(2\pi t))$$

These parameterizations trace out the same curve and we say that they have the same *trace*. Q: why do we need to use parametrized curve?
A: Let's understand the reason by above examples. A parametrized curve describes a curve as a particle, it records the moving particle orientations, even back and forth motion in the middle of a circle or a motion cover the circle many times. But $x^2 + y^2 = 1$ is after the "battle", just leaving us a trace. Therefore parametrized curves can capture dynamics and kinematics in physics and computer vision (e.g. see a person moving forward then backward on pedestrian walkways.)

Consider the magnitude of the derivative of the first parameterization. We find that

$$\begin{aligned} ||\alpha'(t)|| &= \sqrt{(-a\sin(t))^2 + (a\cos(t))^2} \\ &= \sqrt{a^2(\sin^2 t + \cos^2(t))} \\ &= \sqrt{a^2} \\ &= a. \end{aligned}$$

For the other parameterization, γ, we find that

$$\begin{aligned} \gamma'(t) &= (-2\pi a\sin(2\pi t), 2\pi a\cos(2\pi t) \\ ||\gamma'(t)|| &= \sqrt{(-2\pi a)^2(\sin^2 t + \cos^2(t))} \\ &= 2\pi a. \end{aligned}$$

Example 3.

$$\begin{aligned} \alpha\colon \mathbb{R} &\to \mathbb{R}^2 \\ t &\to (t^3, t^2) = \alpha(t) \\ \alpha'(t) &= (3t^2, 2t) = \vec{0} \\ &\begin{cases} 3t^2 = 0 \\ 2t = 0 \end{cases} \quad t = 0 \end{aligned}$$

α at t=0 is not regular.

Example 4.

$$\begin{aligned} \alpha\colon \mathbb{R} &\to \mathbb{R}^2 \\ t &\to (t^3 - 4t, t^2 - 4) = \alpha(t) \\ \alpha'(t) = (3t^2 - 4, 2t) = \vec{0} \Longleftrightarrow &\begin{cases} 3t^2 - 4 = 0 \\ 2t = 0 \end{cases} \quad t = 0, -4 \neq 0. \end{aligned}$$

Every point is regular.

Example 5. See the slides. Example 6. Find the intersection curve in a parametric form of sphere $x^2 + y^2 + z^2 = 2^2$ and cylinder $(x-1)^2 + y^2 = 1$. This will be our homework!

Solution: Hint: Let

$$\begin{cases} x - 1 = \cos(t) \\ y = \sin(t) \\ z = s \end{cases}$$

Then, plug this set of equations into the equation of the sphere. You get

$$(1 + \cos(t))^2 + \sin(t)^2 + s^2 = 4.$$

Then you can solve for s in terms of t. This is called a Viviani curve. If you want to find the lenth of this curve, them the length formula is non-integrable. Our homework will be to find a parametrization of this curve, and approximate the length of this curve use numerical method.

2 Arc Length of a Curve

Definition. Arc length. Refer to handout. Example: consider the circle

$$\alpha(t) = (\cos(t), \sin(t)),$$

where $0 \leq t \leq 2\pi$. Equivalently, consider

$$\beta(t) = (\cos(2\pi t), \sin(2\pi t)),$$

where $t \in [0, 1]$

Definition. α is parameterized by arc length if an only if it has unit speed.

In the above example, α is parameterized by arc length but β is not. That is since

$$\begin{aligned} \alpha'(t) &= (-\sin(t), \cos(t)) \\ ||\alpha'(t)|| &= 1 \end{aligned}$$

while

$$\begin{aligned} \beta'(t) &= (-2\pi \sin(t), 2\pi \cos(t)) \\ ||\beta'(t)|| &= 2\pi && \text{(which is not 1).} \end{aligned}$$

Why do we need a curve parameterized by arc length? Here is why.

$$\begin{aligned} \alpha(s) & && \text{(parameterized by arc length)} \\ ||\alpha'(s)|| &= 1 \\ ||\alpha'(s)||^2 &= 1 \\ \alpha'(s) \cdot \alpha'(s) &= 1 \end{aligned}$$

When you take the derivative of both sides, you get

$$\begin{aligned} \alpha''(s) \cdot \alpha'(s) + \alpha'(s) \cdot \alpha''(s) &= 0 \\ 2(\alpha''(s) \cdot \alpha'(s)) &= 0 \\ \alpha''(s) &\perp \alpha'(s) \end{aligned}$$

Which shows that the second derivative of α is guaranteed to be orthogonal to its velocity if it is parameterized by arc length. This has applications to the Frenet Frame.

Definition. Frenet Frame. With $||\alpha'(s)|| = 1$, let $\vec{t} = \alpha'(s)$ be the tangent vector. Then, let

$$\vec{n} = \frac{\alpha''(s)}{||\alpha''(s)||}.$$

Define $\vec{b} = \vec{t} \times \vec{n}$. Now $\{\vec{t}, \vec{n}, \vec{b}\}$ forms the Frenet Frame, which is an orthonormal frame.

Now taking the derivative of $\vec{b} = \vec{t} \times \vec{n}$, you get torsion, Here is some work showing that.

$$\vec{b}(s) = \vec{t}(s) \times \vec{n}(s).$$

Claim:

$$\vec{b}'(s) = \tau(s)\vec{n}(s).$$

Proof: Key idea: we prove $\vec{b}'(s) \perp \vec{t}(s)$ and $\vec{b}'(s) \perp \vec{b}(s)$; which will force $\vec{b}'(s)\vec{n}(s)$

We start with

$$\vec{b}(s) = \vec{t}(s) \times \vec{n}(s).$$

We take the derivative of both sides to get

$$\vec{b}'(s) = vect'(s) \ \times \vec{n}(s) + \vec{t}(s) \times \vec{n}'(s)$$

The left term of the RHS is zero since $\vec{t}'(s)$ is $\vec{n}(s)$, which implies that $\vec{b}''(s) \perp \vec{t}(s)$.

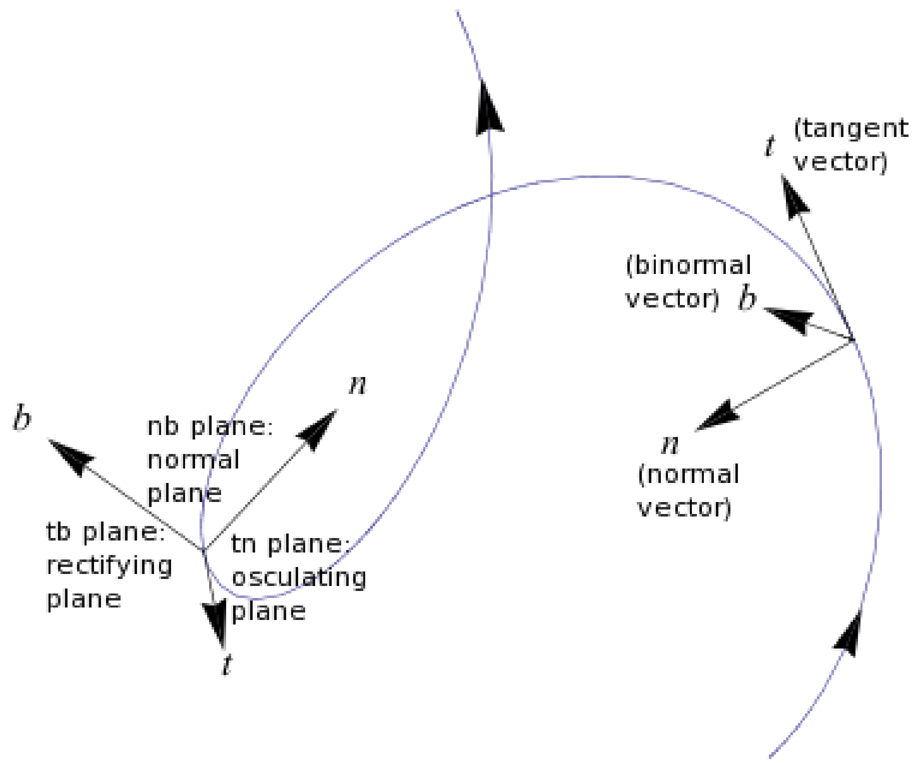
To prove the second claim, we again start with

$$\begin{aligned} \vec{b}(s) &= \vec{t}(s) \times \vec{n}(s) \\ ||\vec{b}(s)|| &= ||\vec{t}(s) \times \vec{n}(s)|| \\ &= ||\vec{t}(s)|| \, ||\vec{n}(s)|| \sin\theta \\ &= 1 \end{aligned}$$

Having shown that, we note the implication that

$$\begin{aligned} ||\vec{b}(s)||^2 &= 1 \\ \vec{b}(s) \cdot \vec{b}(s) &= 1 \qquad \text{(Now take the derivative.)} \\ \vec{b}'(s) \cdot \vec{b}(s) + \vec{b}(s) \cdot \vec{b}'(s) &= 0 \\ 2\vec{b}'(s) \cdot \vec{b}(s) &= 0 \\ \vec{b}'(s) \perp \vec{b}(s) \end{aligned}$$

Note that $\alpha''(s) = (\alpha'(s))'$. That is you can interpret the second derivative of a curve α as the rate of change of its tangent vectors.

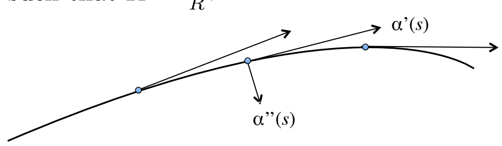


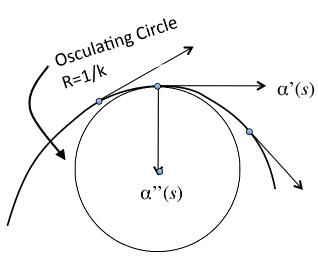
Definition The speed is $\tau(s)$

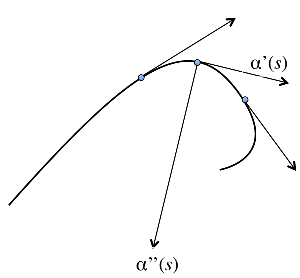
Definition Curvature. Let $\alpha : I \to^3$ be a curve parametrized by arc length $s \in I$. The number $\|\alpha''(s)\| = k(s)$ is called the *curvature* of α at s.

Geometric Meaning Let $\alpha : I = (a, b) \to^3$ be a curve parametrized by arc length s. Since the tangent vector $\alpha'(s)$ has unit length, the norm $\|\alpha''(s)\|$ of the second derivative measures the rate of change of the angle which neighboring tangents make with the tangent at s. $\|\alpha''(s)\|$ gives, therefore, a measure of how rapidly the curve pulls away from the tangent line at s, in a neighborhood of s.

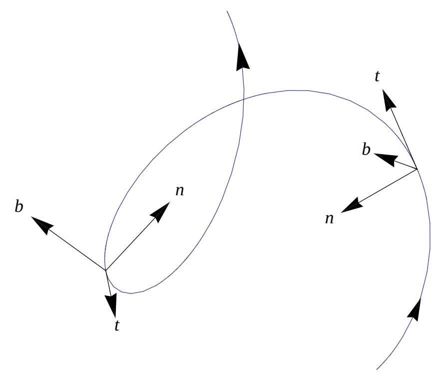
Another geometric meaning: Curvature is related to the radius of the circle most closely fitting a curve at a point, such that $K = \frac{1}{R}$.







Torsion Since $b(s)$ is a unit vector, the length $\|b'(s)\|$ measures the rate of change of the neighboring osculating planes with the osculating plane at s; that is $b'(s)$ measures how rapidly the curve pulls away from the osculating plane at s, in a neighborhood of s.



Let $\alpha : I \to^3$ be a curve parametrized by arc length s such that $\alpha''(s) \neq 0, s \in I$. The number $\tau(s)$ defined by $\vec{b}'(s) = \tau(s)\vec{n}(s)$ is called the *torsion* of α at s.

3 Frenet Formula

$$\begin{cases} \vec{t}'(s) & = 0\vec{t}(s) + k(s)\vec{n}(s) + 0\vec{b}(s) \\ \vec{n}'(s) & = -k(s)\vec{t}(s) + 0\vec{n}(s) + \tau(s) \\ \vec{b}'(s) & = 0\vec{t}(s) + \tau(s)\vec{n}(s) + 0\vec{b}(s) \end{cases}$$

Let's figure out what is $\vec{n}'(s)$.

$$\begin{aligned} \vec{n}(s) &= b(s) \times t(s) \\ \vec{n}'(s) &= b'(s) \times t(s) + b(s) \times t'(s) \\ &= \tau(s)\vec{n}(s) \times \vec{t}(s) + \vec{b} \times k(s)\vec{n}(s) \\ &= -\tau(s)\vec{b}(s) - k(s)\vec{t}(s) \end{aligned}$$

This exercise has shown us that we can fill in the matrix

$$\begin{bmatrix} \vec{t}'(s) \\ \vec{n}'(s) \\ \vec{b}'(s) \end{bmatrix} = \begin{bmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix} \begin{bmatrix} \vec{t}(s) \\ \vec{n}(s) \\ \vec{b}(s) \end{bmatrix}$$

called the Frenet Formula.

This is important because it theoretically gives us a key new method to study curved objects (such as manifolds!) by using a linear algebra problem.

For instance, consider a surface

$$\mathbf{X}(u,v) = (x(u,v), y(u,v), z(u,v)),$$

this could be the sphere $x^2 + y^2 + z^2 = 1$, with parameterization

$$\mathbf{X}(u,v) = (\cos u \cos v, \sin u \cos v, \sin v).$$

Using Frenet's Formula, you can examine the behavior local to any point.

This can be applied to applications such as

- identifying anomalous behavior from UAVs
- determining the trajectory of a cell phone using the onboard sensor data, and using the trajectory to analyze gait and identify users
- robotic surgery

4 Local Canonical Form for Curves

Local canoncical form is an approximation. You get it by taylor expanding around a single point. Consider any curve $\alpha(s)$. Let's expand around 0.

$$\begin{aligned}\alpha(s) &= \alpha(0) + s\alpha'(0) + \frac{s^2}{2!}\alpha''(0) + \frac{s^3}{3!}\alpha'''(0) + \cdots && \alpha(0),\ \alpha'(0) \text{ and th} \\ &= (x(s), y(s), z(s) && \text{Now, Taylor expand each dimension.} \\ &= (x(0) + x'(0)t + \frac{1}{2!}x''(0)t^2 + \dots \quad y(0) + y'(0)t + \frac{1}{2!}y''(0)t^2 + \dots \quad z(0) + z'(0)t + \frac{1}{2!}z''(0)t^2 + \dots) \\ &= \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} + \begin{pmatrix} x'(0) \\ y'(0) \\ z'(0) \end{pmatrix} t + \dots\end{aligned}$$

Recall now that

$$\begin{aligned}\alpha'''(s) &= (\alpha''(s))' \\ &= (k(s)\vec{n}s)' \\ &= k'(s)\vec{n}(s) + \vec{n}'(s)k(s) \\ &= k'(s)\vec{n}(s) + (-k\vec{t} - \tau\vec{b})k(s)\end{aligned}$$

We can similarly investigate the other terms of the Taylor expansion. We use

$$\begin{aligned}\alpha'(s) &= \vec{t}(s) \\ \alpha''(s) &= k(s)\vec{n}(s) \\ \alpha'''(s) &= k'(s)\vec{n}(s) + (-k\vec{t} - \tau\vec{b})k(s)\end{aligned}$$

Let R be the remainder after expanding to 4 terms. which allows us to collect

$$\alpha(s) - \alpha(0) = \left[s - \frac{s^3}{3!}k^2(s)\right]\tau(s) + \left[\frac{s^s}{2!}k(s) + \frac{s^3}{3!}k'(s)\right]\vec{n}(s) + \left[\frac{s^3}{3!}\tau(s)k(s)\right]\vec{b}(s) + \vec{R}$$

which gives us a local representation of the curve, called Local Canonical Form.

$$
\begin{aligned}
x(s) &= s - \frac{s^3}{3!}k^2(s) + R_x \\
y(s) &= \frac{k}{2}s^2 + \frac{s^3}{3!}k'(s) + R_y \\
z(s) &= -\frac{k\tau}{3!}s^3 + R_z
\end{aligned}
$$