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Differential Geometry
Notes
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<u>Review</u>

- $\alpha(s)$ is a regular cuve if $\alpha(s)$ is parameterized by arclength ($||\alpha'(s)||$ = 1)
- $\alpha(s) \perp \alpha''(s)$
- $\vec{t}(s) = \alpha'(s)$
- $\vec{n}(s) = \frac{\alpha''(s)}{||\alpha''(s)||} = \frac{\alpha''(s)}{\alpha(s)}$
  where $k(s) \triangleq ||\alpha''(s)|| = \frac{1}{R(s)}$
- Define $\vec{b}(s) = \vec{t}(s) \times \vec{n}(s)$

<u>Inverse Function Theorem:</u> Monotonic functions are invertable.
<u>Theorem</u> If α is a regular curve in R^3 then there exists a reparameterization β of α such that β has unit speed.
<u>Proof:</u> Let $\alpha : I \to R^3$ be a regular curve Let $s(t) = \int_{t_0}^{t} ||\alpha'(t)||dt$. Then $s'(t) = ||\alpha'(t)||$. Since α is regular, $\alpha'(t) \neq 0$. Then $s'(t) = ||\alpha'(t)|| \neq 0$. Since the derivative never corsses zero and s is continuous the function must be monotonic. Therefore, s has an inverse $t(s)$.

$$\begin{aligned} \frac{dt}{ds} &= \frac{1}{\frac{ds}{dt}} \\ &= \frac{1}{s'(t)} \\ &= \frac{1}{||\alpha'(t)||} \end{aligned}$$

Then $\frac{dt}{ds}$ is always greater than 0. Let β be the reparameeterizatoin,

$$\beta(s) = \alpha(t(s))$$

I calim β has unit speed,

$$\begin{aligned} \beta'(s) &= \alpha'(t(s))t'(s) \\ ||\beta'(s)|| &= ||\alpha'(t(s))||||t'(s)|| \\ &= \cancel{||\alpha'(t(s))||}||\frac{1}{\cancel{\alpha'(t(s))}}|| \\ &= 1 \end{aligned}$$

So β is parameterized by arclength.

Example of this theorem: Consider a helix,

$$\alpha : R \to R^3$$
$$t \mapsto (cost, sint, t) = \alpha(t)$$
$$\alpha'(t) = (-sint, cost, 1)$$
$$||\alpha'(t)|| = \sqrt{(-sint)^2 + (cost)^2 + 1} = \sqrt{2}$$
$$s(t) = \int_0^t ||\alpha'(t)||dt - \int_0^t \sqrt{2}dt = \sqrt{2}t$$
$$\implies t = s/\sqrt{2}$$

So $\beta(s) = (cos\frac{s}{\sqrt{2}}, sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}})$
From now on we can say wihtout loss of generality, we can assume a regualr curve is parameterized by arclength.
Example: A regualr parameterized curve α has the property that all its tangent lines go through a fixed point. a) prove that the trace is a straight line segment.
Proof: Without loss of generality prove that parameterized by arclength.
Let p be the fixed point. A tangent line at $\alpha(s)$ is the line with direction $\alpha'(s)$ and passing throguht the point $\alpha(s)$. The equation for the line is $l(t) = \alpha(s) + t\alpha'(s)$.
By hypothesis, for each choice of s there exists $t(s)$ such that

$$\alpha(s) + \alpha'(s)t(s) = p$$

Notice $t(s)$ is differentiable,

$$\alpha(s) + \alpha'(s)t(s) = p$$
$$\implies \alpha'(s) \cdot \alpha(s) + \alpha'(s) \cdot \alpha'(s)t(s) = p \cdot \alpha'(s)$$
$$\implies t(s) = \frac{\cdot\alpha'(s) - \alpha'(s) \cdot \alpha(s)}{||\alpha'(s)||^2}$$

Since α iks regular, $||\alpha'(s)|| \neq$ so this is a valid expression and $t(s)$ is differentiable. So then we take the derivative of both sides,

$$\alpha(s) + \alpha'(s)t(s) = p$$
$$\implies \alpha'(s) + \alpha''(s)t(s) + \alpha'(s)t'(s) = 0 \qquad \text{(take derivative of both sides)}$$

There are application to UAV autonomous vehicles and cellphones Definition $\{v_1, v_2, v_3\}$ is right hand sided if and only if $det([v_1\ v_2\ v_3]) > 0$
Recall the definition of a group A group G is a finite or infinite set of elements together with a binary operation (called the group operation) that together satisfy the four fundamental properties of closure, associativity, the identity property, and the inverse property.

The operation with respect to which a group is defined is often called the ”group operation,” and a set is said to be a group ”under” this operation. Elements A, B, C, ... with binary operation between A and B denoted AB form a group if

1. Closure: If A and B are two elements in G, then the product AB is also in G.

2. Associativity: The defined multiplication is associative, i.e., for all A,B,C in G, $(AB)C = A(BC)$.

3. Identity: There is an identity element I (a.k.a. 1, E, or e) such that $IA = AI = A$ for every element A in G.

4. Inverse: There must be an inverse (a.k.a. reciprocal) of each element. Therefore, for each element A of G, the set contains an element $B = A^{-1}$ such that $AA^{-1} = A^{-1A} = I$.

<u>Claim:</u> The set of all 3×3 orthonormal matrices forms a group denoted $O(3)$:

$$O(3) = \{A \in M_{3\times3}(R) : A^T A = I\}$$

Then there is $SO(3) = \{A \in O(3) : detA = 1\} < O(3)$. Define he matrix multiplicationi operator as

$$\begin{aligned} O(3) \times O(3) &\to O(3) \\ A, B &\mapsto AB \\ A \in O(3) \implies (A^T TA) &= IanddetA = 1 \\ B \in O(3) \implies B^T B &= IanddetB = 1 \end{aligned}$$

Now we show $AB \in O(3)$,

$$(AB)^T (AB) = B^T A^T AB = B^T B = I \in O(3)$$

So it is. Now we show that $A^{-1} \in O(3)$.

$$A^{-1}(A^{-1})^T = A^{-1}A = I$$