Joseph Gardi
Differential Geometry
Notes
Monday, September 30 2019

Preliminary Definitions

- A neighborhood around a point p is $\{x : ||x-p|| \leq r\}$ for some radius r. This set is shaped like a ball and denotoed $B_r(p)$.
- If we say something is true locally around x we mean it is true for all points in some sufficiently small neighborhood around x.
- A homeomorphism is a continuous function with a coninuous inverse.
- Intuition for continuous funcions. If f is continuous then a close to b implies that $f(a)$ close $f(b)$. To illustrate consider my body. My wrist is close to my hand. So if we apply a continuous transformationsto the position of each of my body parts theen my wrist will still be close to my hand. All the movements I normally do are continuous transformations. But if I cut off my own hand then my hand won't be close to my wrist anymore. That is a discontinuous transformation. There is a discontinuous jump from my wrist after cutting to my hand after cutting.
- The differential matrix for a function $f(x,y)$:

$$Df \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

It will also be helpful to remember the first order taylor expansion for a continuous differentiable function $f : R^2 \to R^2$,

$$f(x,y) = f(u,v) + Df(u,v) \begin{bmatrix} x-u \\ x-v \end{bmatrix}$$

Theorem: Inverse function theorem in high dimension
Let $\rho : (u,v) \to (x(u,v), y(u,v))$ be a differentiable function.
ρ is locally (i.e. for some sufficiently small neighborhood) invertable around (a,b) if and only if $D\rho((a,b))$ is an invertable matrix. Recall,

$$D\rho = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Also, ρ^{-1} is differentiable.

We will first define regular surface. Then we take characteristic properties of regular surfaces to define manifolds. Manifolds are like a generalization of regular surfaces.
We will put the reimennian matric on each tangent space.

This is similar to how we generalized R^3 to metric spaces in analysis.

Suppose we are interested in the miotion of a particle. Then we take, as the state oof the particle, the pair of 3 dimensional vectoros (x, v) where x is the position and v is the velocity. If we know the particle must stay on a sphere M then we it follows that v must always be tangent to M. Then our state space S is not all pairs of 3-vectors but he tangent bundle of M which is a manifold.

$$S = \{(x, v) : v \text{ is tangent to } M\}$$

Definition: $S \subset R^3$ is a regular surface if for each $p \in S$ there exists a neighborhood V in R^3 and a map $x : U \to V \bigcap S$ of an open set $U \subset R^2$ onto $V \bigcap S \subset R^3$ such that,

- x is differentiable (so we can use calculus)
- x is a homeomoorphism (so we can use analysis)
- x is regular (so we can use linear algebra). Since x is regular there is a tangent plane at each point in S.

Suppose we have a regular surface $V \subseteq R^3$ with a mapping $x : (u, v) \to (x(u, v)y(u, v), z(u, v))$. Then the tangent plane at a point q is spanned by the columns of $Dx_q = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}$.

Notation:

$$\frac{\partial(x, y)}{\partial(u, v)} \triangleq \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial u}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$\{(x, y, f(x, y)) : x, y \in R\}$ is a surface.
The parameterization for this surface is $a(x, y) = (x, y, f(x, y))$

Now we willl prove his is a regular surface. $da = \begin{bmatrix} 1^0 & \\ 0 & 1 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$. This is invertable so a has a continuous inverse. So it is homeomorphic. Definition: For a differentible function $f : U \subset R^3 R$.