Lecture 14: Christoffel Symbols and the Compatibility Equations

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Math 142:
Differential Geometry

Christoffel Symbols

Trihedron at a Point of a Surface

S will denote, as usual, a regular, orientable, and oriented surface. Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ be a parametrization in the orientation of S. It is possible to assign to each point of $\mathbf{x}(U)$ a natural trihedron given by the vectors $\mathbf{x}_u$, $\mathbf{x}_v$, and N.

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By expressing the derivatives of the vectors $\mathbf{x}_u$, $\mathbf{x}_v$, and N in the basis $\{\mathbf{x}_u, \mathbf{x}_v, N\}$, we obtain

$$\begin{aligned}
\mathbf{x}_{uu} &= \Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + L_1 N, \\
\mathbf{x}_{uv} &= \Gamma^2_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + L_2 N, \\
\mathbf{x}_{vu} &= \Gamma^1_{21}\mathbf{x}_u + \Gamma^2_{21}\mathbf{x}_v + \overline{L}_2 N, \\
\mathbf{x}_{vv} &= \Gamma^1_{22}\mathbf{x}_u + \Gamma^2_{22}\mathbf{x}_v + L_3 N, \\
N_u &= a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \\
N_v &= a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v.
\end{aligned}$$

Christoffel Symbols

Note

By taking the inner product of the first four relations on the previous slide with N, we immediately obtain $L_1 = e$, $L_2 = \overline{L}_2 = f$, $L_3 = g$, where e, f, and g are the coefficients of the second fundamental form of S.

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Note

The a_{ij}'s in the last two relations on the previous slides come from the matrix representation

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

of dN_p.

Christoffel Symbols

Definition

The coefficients Γ_{ij}^{k}, $i, j, k = 1, 2$, are called the *Christoffel symbols* of S in the parametrization $\mathbf{x}$. Since $\mathbf{x}_{uv} = \mathbf{x}_{vu}$, we conclude that $\Gamma_{12}^{1} = \Gamma_{21}^{1}$ and $\Gamma_{12}^{2} = \Gamma_{21}^{2}$; that is, the Christoffel symbols are symmetric relative to the lower indices.

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To determine the Christoffel symbols, we take the inner product of the first four relations with $\mathbf{x}_u$ and $\mathbf{x}_v$, obtaining the system

$$
\begin{cases}
\Gamma_{11}^1 E + \Gamma_{11}^2 F = \langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle = \frac{1}{2} E_u, \\
\Gamma_{11}^1 F + \Gamma_{11}^2 G = \langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle = F_u - \frac{1}{2} E_v,
\end{cases}
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\Gamma_{12}^1 E + \Gamma_{12}^2 F = \langle \mathbf{x}_{uv}, \mathbf{x}_u \rangle = \frac{1}{2} E_v, \\
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\Gamma_{22}^1 E + \Gamma_{22}^2 F = \langle \mathbf{x}_{vv}, \mathbf{x}_u \rangle = F_v - \frac{1}{2} G_u, \\
\Gamma_{22}^1 F + \Gamma_{22}^2 G = \langle \mathbf{x}_{vv}, \mathbf{x}_v \rangle = \frac{1}{2} G_v.
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Christoffel Symbols

Note: $EG - F^2 \neq 0$

Thus, it is possible to solve the above system (use Cramer's Rule) and *to compute the Christoffel symbols in terms of the coefficients of the first fundamental form, E, F, G, and their derivatives.*

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Important Observation

All geometric concepts and properties expressed in terms of the Christoffel symbols are invariant under isometries.

Christoffel Symbols

Example

We shall compute the Christoffel symbols for a surface of revolution parametrized by

$$\mathbf{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v)), \qquad f(v) \neq 0.$$

Recall

$$E = (f(v))^2 \neq 0, \qquad F = 0, \qquad G = (f'(v))^2 + (g'(v))^2 \neq 0.$$

The Theorem of Gauss

Theorem (Theorema Egregium (Gauss))

The Gaussian curvature K of a surface is invariant by local isometries.

Proof.

$\Longrightarrow$

$$(\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12} = -E\,\frac{eg - f^2}{EG - F^2} = -EK. \qquad (1)$$

Theorem of Gauss

Consequences

- In fact, if $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ is a parametrization at $p \in S$ and if $\varphi : V \subset S \to S$, where $V \subset \mathbf{x}(U)$ is a neighborhood of p, is a local isometry at p, then $\mathbf{y} = \varphi \circ \mathbf{x}$ is a parametrization of S at $\varphi(p)$.

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- Since φ is an isometry, the coefficients of the first fundamental form in the parametrizations $\mathbf{x}$ and $\mathbf{y}$ agree at corresponding points q and $\varphi(q)$, $q \in V$; thus, the corresponding Christoffel symbols also agree.

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- Since φ is an isometry, the coefficients of the first fundamental form in the parametrizations $\mathbf{x}$ and $\mathbf{y}$ agree at corresponding points q and $\varphi(q)$, $q \in V$; thus, the corresponding Christoffel symbols also agree.
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- Since φ is an isometry, the coefficients of the first fundamental form in the parametrizations $\mathbf{x}$ and $\mathbf{y}$ agree at corresponding points q and $\varphi(q)$, $q \in V$; thus, the corresponding Christoffel symbols also agree.
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Example

Recall that a catenoid is locally isometric to a helicoid. It follows from the Gauss theorem that the Gaussian curvatures are equal at corresponding points, a fact which is geometrically nontrivial.

Importance of Gauss's Formula

Gauss's Formula

$$K = -\frac{1}{E}\left[\left(\Gamma_{12}^2\right)_u - \left(\Gamma_{11}^2\right)_v + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{11}^1\Gamma_{12}^2\right].$$

When $\mathbf{x}$ is an orthogonal parametrization (i.e., $F = 0$), then

$$K = -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right].$$

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Why is this cool?

The Gauss formula expresses the Gaussian curvature K as a function of the coefficients of the first fundamental form and its derivatives. This means that K is an intrinsic concept, a very striking fact if we consider that K was defined using the second fundamental form.

Intrinsic Geometry

We shall soon see that many other concepts of differential geometry are in the same setting as the Gaussian curvature; that is, they depend only on the first fundamental form of the surface. It thus makes sense to talk about a geometry of the first fundamental form, which we call intrinsic geometry, since it may be developed without any reference to the space that contains the surface (once the first fundamental form is given).

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Example

The Mainardi-Codazzi Equations are similar to the Gauss formula, and are given by

$$f_v - g_u = e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12}$$
$$e_v - f_u = e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11}.$$

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- The Gauss formula and the Mainardi-Codazzi equations are known under the name of *compatibility equations* of the theory of surfaces.
- A natural question is whether there exist further relations of compatibility between the first and second fundamental forms besides those already obtained.

Bonnet's Theorem

Theorem (Bonnet)

Let E, F, G, e, f, g *be differentiable functions, defined in an open set* $V \subset \mathbb{R}^2$*, with* $E > 0$ *and* $G > 0$*. Assume that the given functions satisfy formally the Gauss and Mainardi-Codazzi equations and that* $EG - F^2 > 0$*. Then, for every* $q \in V$ *there exists a neighborhood* $U \subset V$ *of* q *and a diffeomorphism* $\mathbf{x} : U \to \mathbf{x}(U) \subset \mathbb{R}^3$ *such that the regular surface* $\mathbf{x}(U) \subset \mathbb{R}^3$ *has* E, F, G *and* e, f, g *as coefficients of the first and second fundamental forms, respectively. Furthermore, if* U *is connected and if*

$$\bar{\mathbf{x}} : U \to \bar{\mathbf{x}}(U) \subset \mathbb{R}^3$$

is another diffeomorphism satisfying the same conditions, then there exists a translation T *and a proper linear orthogonal transformation* ρ *in* $\mathbb{R}^3$ *such that* $\bar{\mathbf{x}} = T \circ \rho \circ \mathbf{x}$.